Lecture 1. Fluid mechanics as a science

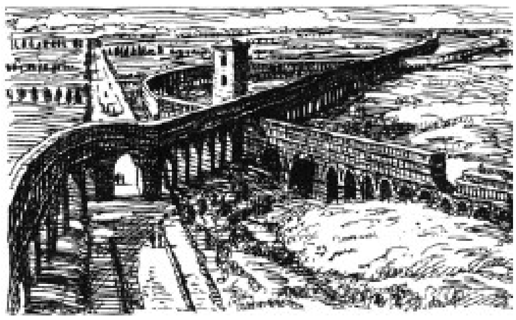


Figure 1.1 Restored arch of Roman aqueduct in Compania Plain, Italy

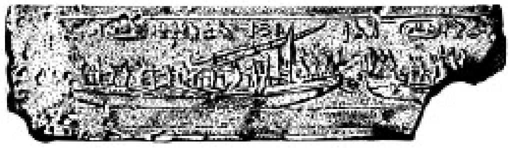


Figure 1.2 Relief of ancient Egyptian ship

Figure 1.3 Ancient Greek ship depicted on old vase

Figure 1.4 Leonardo da Vinci (1452-1519)

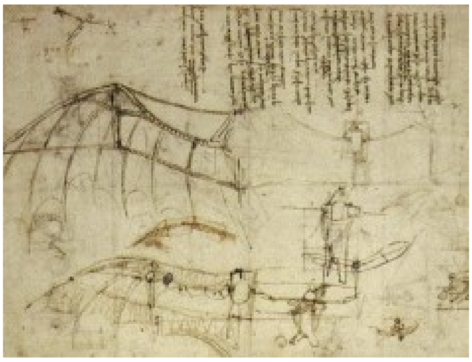


Figure 1.5 Ornithopter' wings of Leonardo da Vinci

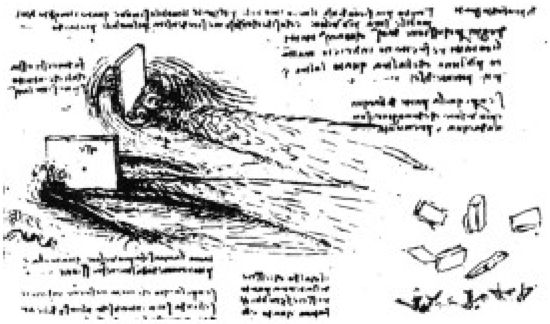


Figure 1.6 Sketches from Leonardo da Vinci's notes

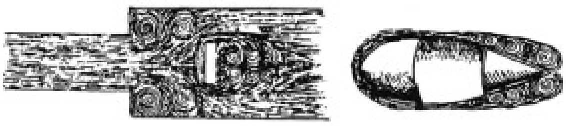


Figure 1.7 Sketches from Leonardo da Vinci's notes

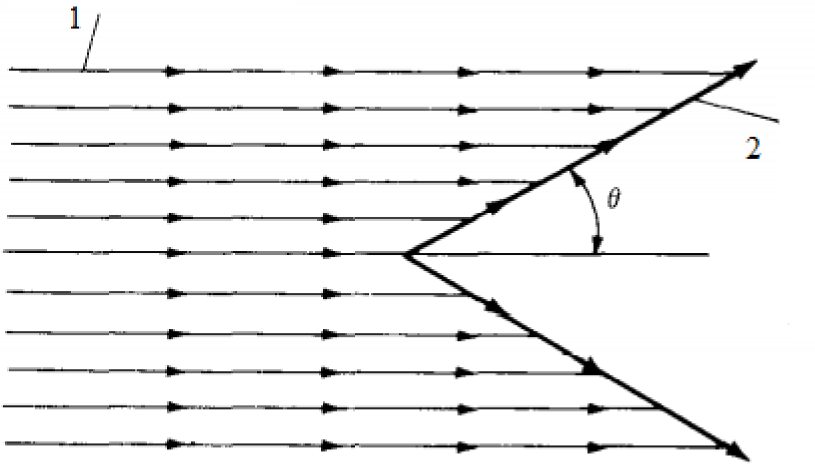


Figure 1.8 Isaac Newton's model of fluid flow in the year 1687. This model was widely adopted in the seventeenth and eighteenth centuries but was later found to be conceptually inaccurate for most fluid flows:
1 – rectilinear stream of descrete particles; 2 – upon impacting the body, the partiles give up their momentum normal to the surface, and travel downstream along the surface.

Control questions
1. What do you know about fluid mechanics in everyday life?
2. What was the beginning of fluid mechanics?
3. How did fluid mechanics begin to take shape as a science in the middle of the 15th century?
4. What contribution did I. Newton to fluid mechanics?
5. What contribution did Euler and Bernoulli make to fluid mechanics?
6. What contribution did ME Zhukovsky, S.O. Chaplygin and DI Mendeleev in fluid mechanics?
7. What do you know about laminar and turbulent flow?
8. What do you know about the development of the doctrine of vortex motions?
9. What components are divided fluid mechanics on?
10. What is the principle of conservation of motion for a fluid?
11. What is the connection between the theory of fluid mechanics and experiment?
12. How is the knowledge of fluid mechanics applied in the design of flying machines?

Lecture 2. The Relevant Properties of a Fluid

$$\frac{\frac{-1}{v}\Delta v}{\Delta p}=\frac{-1}{v}\frac{\partial v}{\partial p}.$$

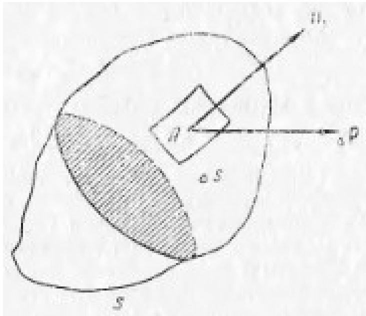


Figure 2.1 A certain volume occupied the space by the fluid

$$\beta=\frac{-1}{\rho}\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial p}=\frac{-1}{\rho}\frac{\partial \rho}{\partial \rho}.$$

$$E=\frac{1}{\beta_v}=-v\frac{\partial p}{\partial v}=\rho\frac{\partial p}{\partial \rho}$$

$$E = 1 / \beta_v = p.$$

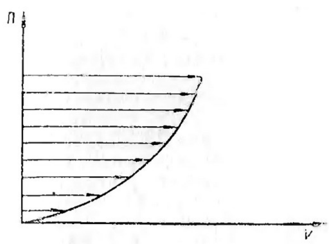


Figure 2.2 For calculation of internal friction tangential stresses

$$\tau = \mu \; dv/dn$$

$$v = \mu / \rho.$$

$$\mu=\mu_0\frac{1+\frac{C}{273}}{1+\frac{C}{273+\theta}}\sqrt{\frac{\theta+273}{273}},$$

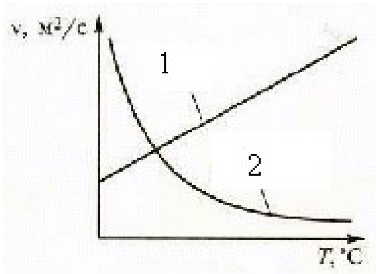


Fig. 2.3 Viscosity & temperature:
1 - drip liquid, 2 - gas

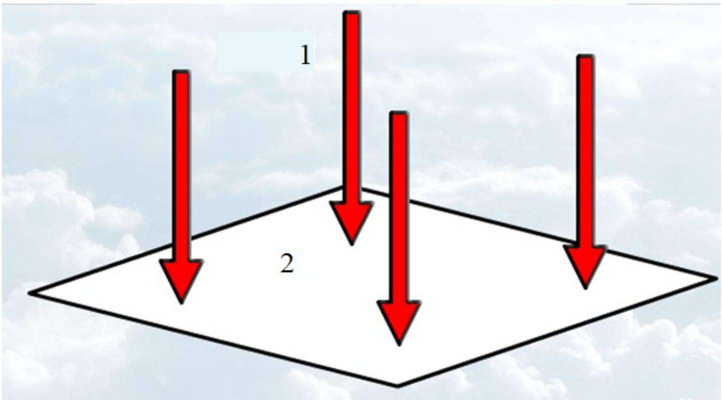


Figure 2.4. To determine pressure
1 - force; 2 - area

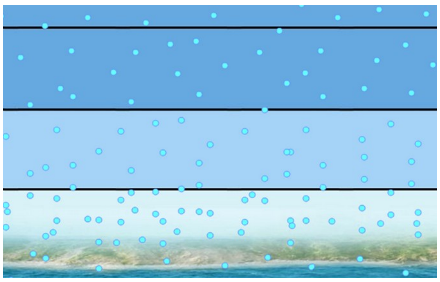


Fig. 2.5. Rarefaction of air with height

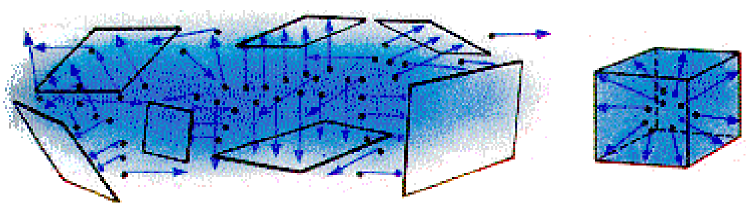


Fig. 2.6. Pressure

$$p = P/A$$

$$\frac{\Delta P}{\Delta A} = \frac{dP}{dA}$$

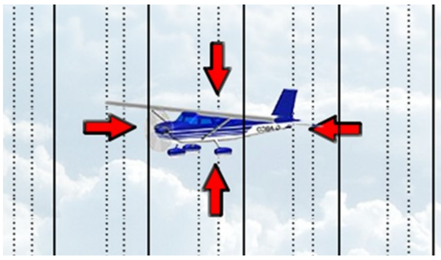


Fig. 2.7. Uniform transfer of static pressure in all directions

Table 2.1 Conversion of pressure units

Name of unit	Unit	Conversion
Pascal	Pa	1 Pa = 1 N/m^2
Bar	bar	1 bar = 0.1 MPa
Water column metre	mH_20	1 m H_20 = 9 806.65 Pa
Atmospheric pressure	atm	1 atm = 101 325 Pa
Mercury column metre	mHg	1 mHg = 1/0.76 atm
Torr	torr	1 torr = 1 mm Hg

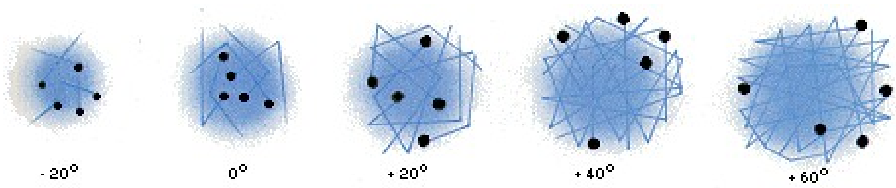


Fig. 2.8. Temperature

$$\rho = m/V.$$

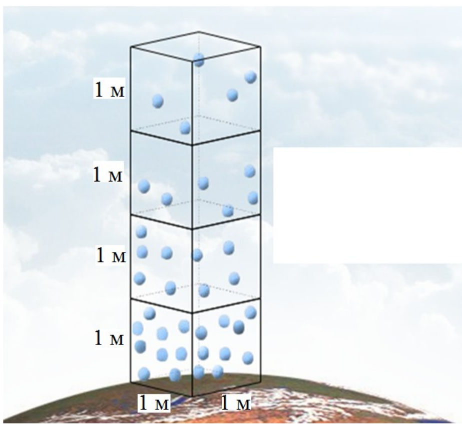


Fig. 2.9. To determine the density

$$\gamma = G/V.$$

$$G = mg: \gamma = G / V = mg / V = \rho\ g.$$

$$s = \rho/\rho_w.$$

$$v = 1/\rho$$

ρ_0: $\Delta\rho = \rho_H / \rho_0$.

$$\rho = \frac{p}{gRT},$$

$$[R] = [L^2\ T^{-2}\ \theta^{-1}],$$

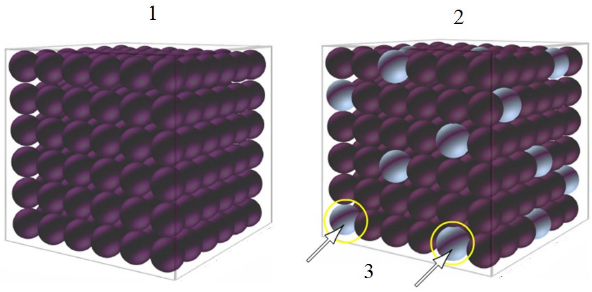


Fig. 2.10. Influence of humidity on air density: 1 - dry air (high density), 2 - humid air (lower density), 3 - water vapor

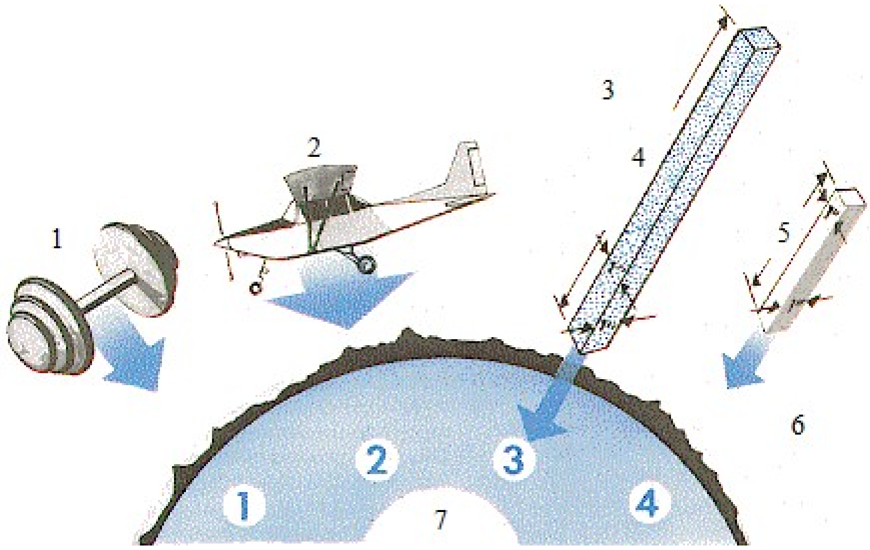


Fig. 2.11 Weight has a direction and is measured in Newtons:
1 - rods: 20 n, 2 - aircraft: 250 n, 3 - air: 1 " x 1 " x 60 miles = 1.47 n, 4 - 60 km, 5 - 29.92 ", 6 - mercury: 1 " x 1 " x 29.92 " = 1.47 n, 7 - the center of the Earth

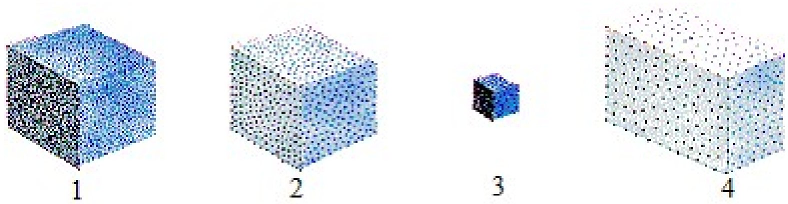


Fig. 2.11 Weight:
1 - 30 kg, 2 - 20 kg, 3 - 10 kg, 4 - 10 kg

It is accepted as initial conditions for the standard atmosphere:

• for the troposphere (at μ = 0.0065 deg/m) H_0 = 0 (from sea level), T = 288 K, p_0 = 1.013 MPa, ρ_0 = 1.225 kg / m3;

• for the stratosphere (at μ = 0 deg/m) H_0 = 11000 m, T = 216 K = const.

1. Troposphere (for altitudes up to 11 km);

$$p/p_0=(1-H/44300)^{5,256};\qquad \Delta=\rho/\rho_0=(1-H/44300)^{4,256};$$

$$\rho=1,225\,(1-H/44300)^{4,256}.$$

2. Stratosphere (for altitudes over 11 km).

$$p=0,0371\,e^{-\frac{H-1100}{6340}}$$

Control questions

1. In what three main forms can matter exist? What causes these forms?
2. What is a fluid from a mechanical point of view? What laws does fluid obey?
3. What does the assumption of mass conservation and continuum mean?
4. What is the essence of the continuity hypothesis?
5. What are the basic physical properties of a liquid?
6. What is it compressibility and what is a measure of compressibility?
7. How are calculated the tangential stresses?
8. What is viscosity and how does viscosity depend on temperature?
9. What is it pressure and from what factors and how it depends?
10. What is it such a physical quantity as temperature?
11. What is it the dencity of a substance?
12. What is it the International Standard Atmosphere and why is it needed?

Lecture 3. Fundamentals of hydrostatics

$$\tau = T / S.$$

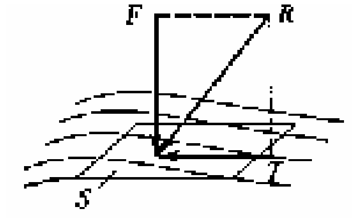


Fig. 3.1 Scheme of action of surface forces

$$p = F/S. \quad (3.1)$$

$$p_1 dA_1 = p dA \sin\theta$$
$$p_2 dA_2 = p dA \cos\theta + \tfrac{1}{2}\, dA_1 dA_2 pg$$

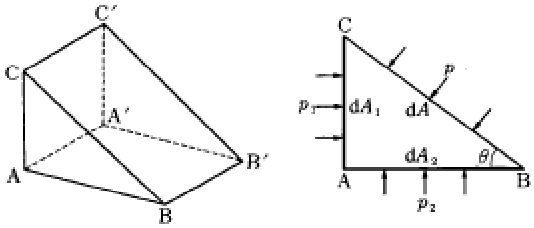


Fig. 3.2 Pressure acting on a minute triangular prism

$$dA \sin\theta = dA_1$$
$$dA \cos\theta = dA_2$$

$$p_1 = p_2 = p$$

Fig. 3.3 Blaise Pascal (1623-62)
French mathematician, physicist and philosopher. He had the ability of a highly gifted scientist even in early life, invented an arithmetic computer at 19 years old and discovered the principle of fluid mechanics that carries his name. Many units had appeared as the units of pressure, but it was decided to use the pascal in SI units in memory of his achievements.

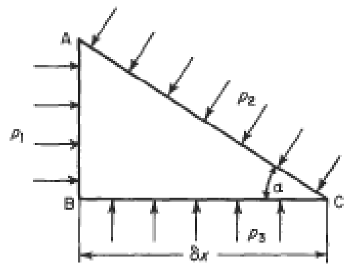


Fig. 3.4 The prism for Pascal's Law

$$p_1(\delta x \tan\alpha)\delta z - p_2(\delta x \sec\alpha)\delta z \sin\alpha = 0$$

$$p_1 - p_2 = 0$$

$$p_1 = p_2$$

$$p_3 \delta x \delta z - p_2(\delta x \sec\alpha)\delta z \cos\alpha - W = 0$$

$$W = \rho g(\delta x)^2 \tan\alpha \delta z / 2$$

$$p_3 - p_2 - \frac{1}{2}\rho g \tan\alpha \delta z = 0$$

$$p_3 - p_2 = 0$$

$$p_1 = p_2 = p_3$$

$$F_2 = F_1 A_2 / A_1$$

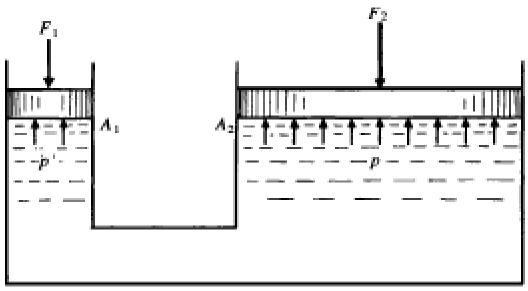


Fig. 3.5 Hydraulic press

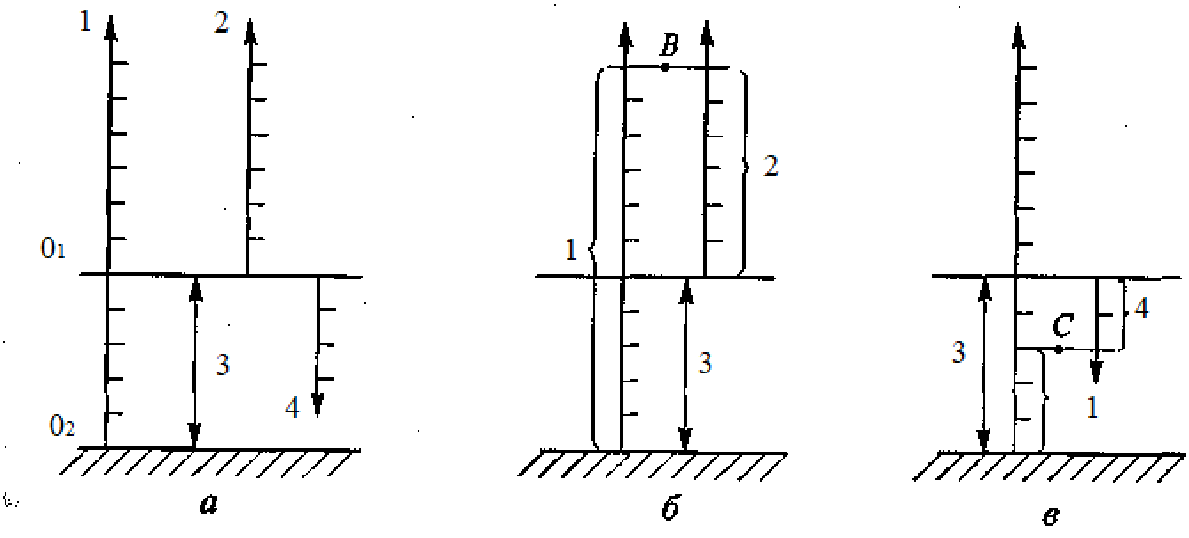


Fig. 3.6 Pressure reference systems:
a - pressure scales; b - relationship of absolute and excess pressures; в - relationship of absolute pressure and vacuum pressure
1 — P_{abs}; 2 — P_{exc}; 3 — P_a; 4 — P_{vac}; 0_1 — 0_{atm}; 0_2 - 0_{abs}

$$p_{abs} = p_a + p_{exc}, \quad (3.2)$$
$$p_{abs} = p_a - p_{vac} \quad (3.3)$$
$$p_{exc} = - p_{vac}. \quad (3.4)$$

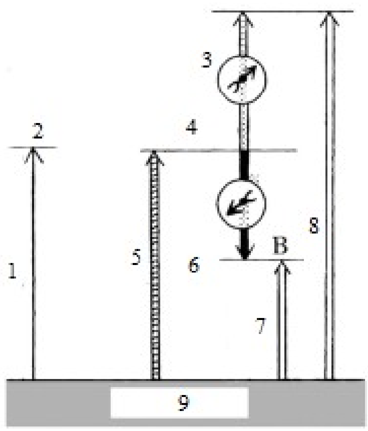


Fig. 3.7 Absolute pressure and gauge pressure:
1 – 760 mm Hg; 2 – 1 atm; 3 – Gauge pressure (+); 4 – Gauge pressure 0; 5 – Barometer reading; 6 – Gauge pressure (-); 7 – Absolute pressure; 8 – Absolute pressure; 9 – Perfect vacuum

$$p_{абс} = p_a + \rho\, gh_p$$

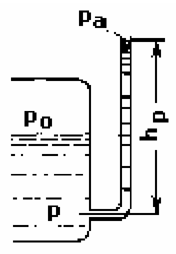


Fig. 3.8 Measurement of excess pressure by a manometer

$$h_p – (p_{абс} - p_a)/\rho g = p_{над}/\rho g.$$

$$p + \rho gH = p_0 +\rho' gH'$$

$$p = p_0 +\rho' gH' -\rho gH \quad (3.5)$$

$$p_1 - p_2 = (\rho- \rho' gH)$$

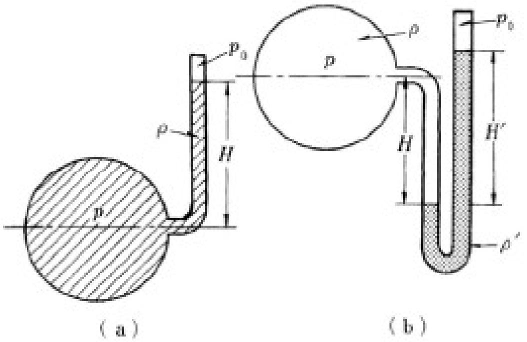


Fig. 3.9 Manometer

$$\rho_1 - \rho_2 = \rho gH$$

$$p_1 - p_2 = (\rho'-\rho)gH'$$

$$p_1 - p_2 =\rho' gH'$$

$$p_b +\rho gh_1 = p_a +\rho gh_2$$

$$p_b = p_a -\rho g(h_1-h_2)$$

$$p_b = p_a -\rho g\Delta h$$

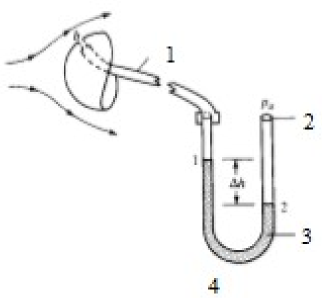


Figure 3.10 The use of a U-tube manometer:
1 – flexible pressure tube; 2 – tube open to atmosphere; 3 - liquid with density ρ (frequently mercury or silicone oil); 4 – U-tube manometer (usually made from glass tubing)

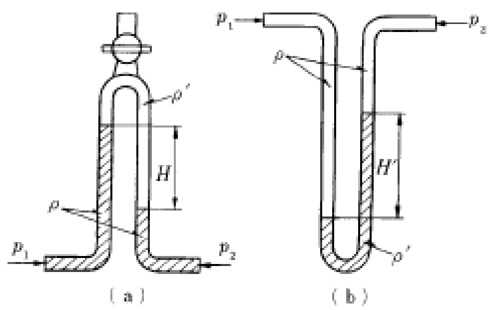


Fig. 3.11 Differential manometer

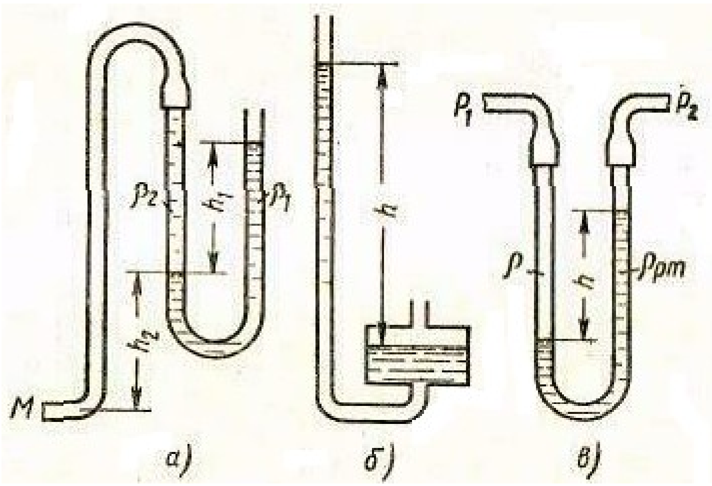


Fig. 3.12 Diagrams of fluid gauges:
a) U-shaped manometer; b) cup gauge; c) differential pressure gauge.

$$H = L\sin\alpha$$

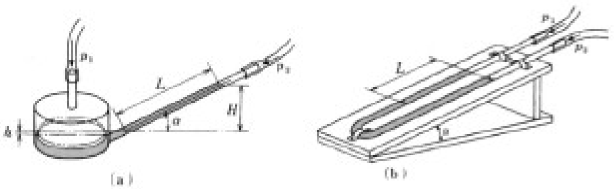


Fig. 3.13 Inclined manometer

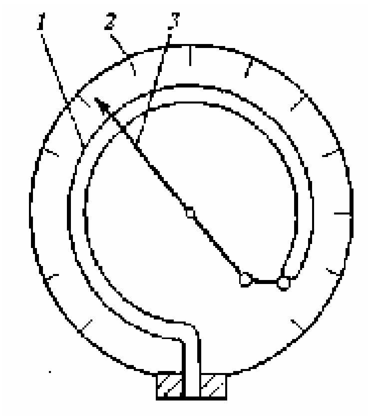


Fig. 3.14 Spring pressure gauge:
1 - tube, 2 - scale, 3 - arrow

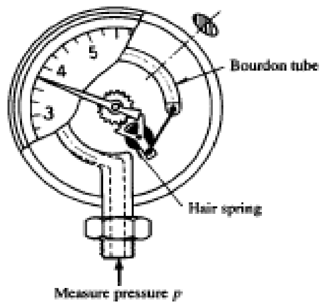


Fig. 3.15 Bourdon tube pressure gauge

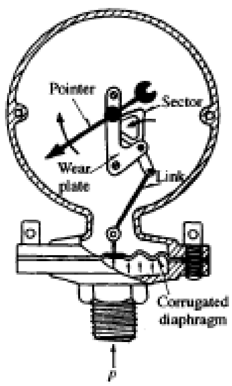
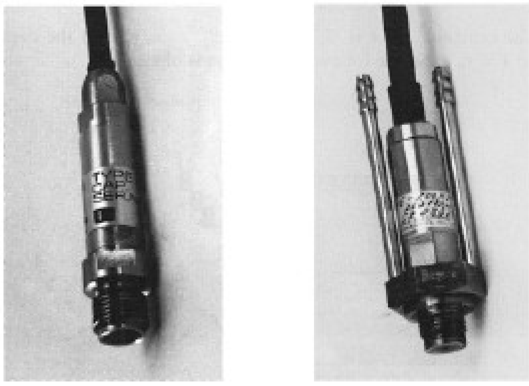


Fig. 3.16 Diaphragm pressure gauge



	PGM-C	PE-J, P
	(a) Small size	(b) For indicating pressure of an engine
Pressure range (MPa)	0.2–1.0	5–20
Natural frequency (kHz)	25–40	25–40
Response frequency (kHz)	5–8	5–8

Fig. 3.17 Wire strain gauge type of pressure transducer

Control questions

1. What is the subject of hydraulics? What are the basic concepts and definitions of hydraulics?
2. What are the two methods of research and solution of technical problems of hydraulics?
3. What issues are considered in hydrostatics?
6. What are the pressure counting?
7. What are the properties of hydrostatic pressure?
8. What devices are used to measure pressure?
9. What are the schemes of the most common liquid manometers and vacuum gauges?
10. What are the features of pressure measuring with manometers?

Lecture 4. The basic equation of hydrostatics.

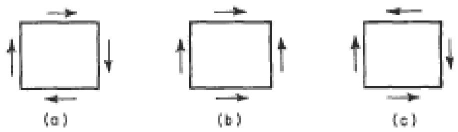


Fig. 4.1 Fictitious systems of tangential forces in static fluid

pdA — ($p + dp/dz\ dz$) dA — $\rho g dA dz = 0$

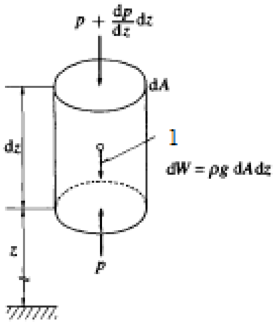


Fig. 4.2 Balance of vertical minute cylinder:
1 - Weight

$$dp/dz = -\rho g \quad (4.1)$$

$$p = -\rho g \int dz = -\rho g z + c \qquad (4.2)$$

$$c = p_0 + \rho g z$$

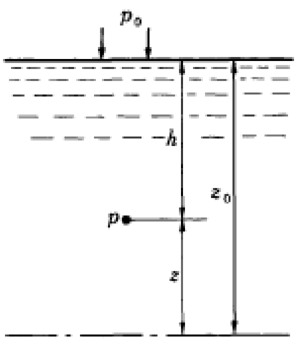


Fig. 4.3 Pressure in liquid

$$p = p_0 + (z_0 - z)\,\rho g = p_0 + \rho g h$$

$$p/\rho^n = p_0/\rho^n_0 \quad (4.3)$$

$$dz = -dp/g = -1/g\ p_o^{1/n}/\rho_o\ p^{-1/n}dp = 1/g\ p_o/\rho_o(p_o/\rho_o)^{1/n}d(p/\rho_o) \quad (4.4)$$

$$z = \int_0^z dz = 1/g\ n/(n-1)\ p_0/\rho_0[1-(p/p_0)^{(n-1)/n}] \quad (4.5)$$

$$p(z)/p_0 = (1-(n-1)/n\ \rho_0 g/p_0\ z)^{(n-1)/n} \quad (4.6)$$

$$p\rho(z)/\rho_0 = (1-(n-1)/n\ \rho_0 g/p_0\ z)^{1/(n-1)} \quad (4.7)$$

$$\frac{p}{\rho T} = \frac{p_0}{\rho_0 T_0} = R \quad (4.8)$$

$T(z)/T_o = 1-(n-1)/n\ \rho_0 g/p_0\ z$ (4.9)

$dT/dz = -(n-1)/n\ \rho_0 g/p_0\ T_o = -(n-1)/n\ g/R$

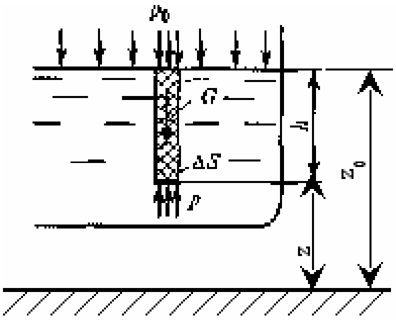


Fig. 4.4 Scheme for obtaining the basic hydrostatics equation

$p\ \Delta S - G - p_0 \cdot \Delta S = 0.$

$G = V \cdot p\ g = \Delta S\ h\ \rho\ g.$

$p = p_0 + \rho\ g\ h.$ (4.10)

$z+p/\rho g=z_0+p_0/\rho g.$

$z+p/\rho g=const$

$p_o = p - \rho\ g\ h.$

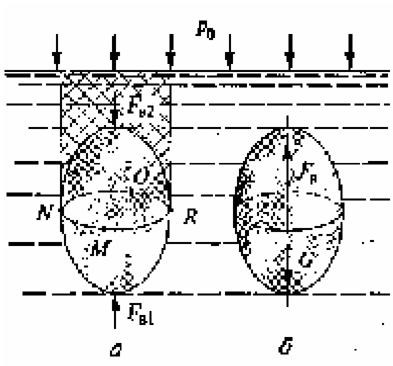


Fig. 4.5 Body Swimming Scheme:
a - to determine the Archimedean force, b - an example of a stable body position

$F_B = p_o\ S_\Gamma + G$:

$F_{B1} = p_0\ S_\Gamma + G_0 + G$, (4.11)

$F_{B2} = p_0\ S_\Gamma + G_0$. (4.12)

$F_a = F_{B1} - F_{B2} = G.$

Archimedes (287-212 BC)
The greatest mathematician, physicist and engineer in ancient Greece, and the discoverer of the famous 'Principle of Archimedes'. Archimedes received guidance in astronomy from his father, an astronomer, and made astronomical observations since his early days. He invented a planetarium turned by hydropower and a screw pump. He carried out research in solid and fluid dynamic as well as on the lever, the centre of gravity and buoyancy. Archimedes was one of those scientists who are talented in both theory and practice.

The Archimedes force formula $F = \rho g V$.

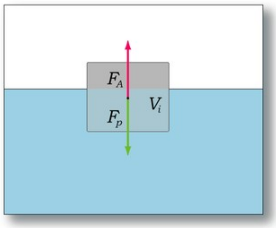


Fig. 4.6 Equilibrium of a body floating on a liquid surface

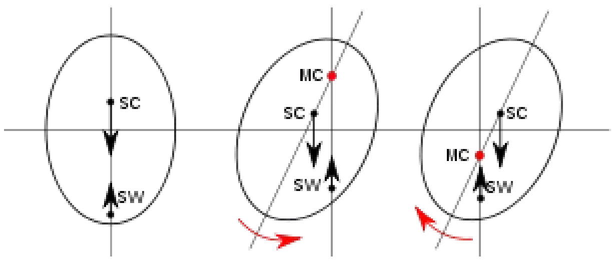


Fig. 4.7 Behavior of a body partially immersed in a liquid

$$W = V_H / V_O \ 100\ \%,$$

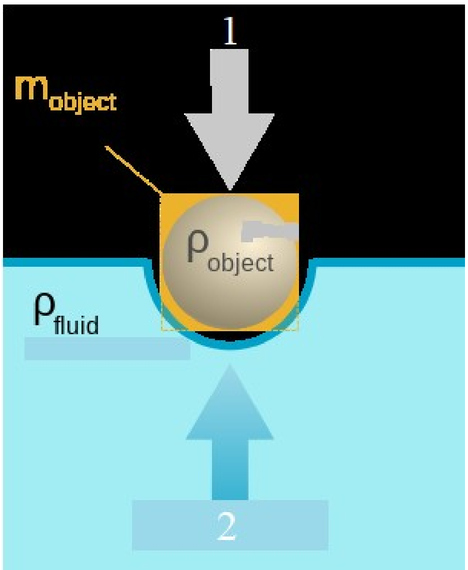


Fig. 4.9 Forces acting on a partially submerged body
1 — Gravity; 2 - Buoyancy

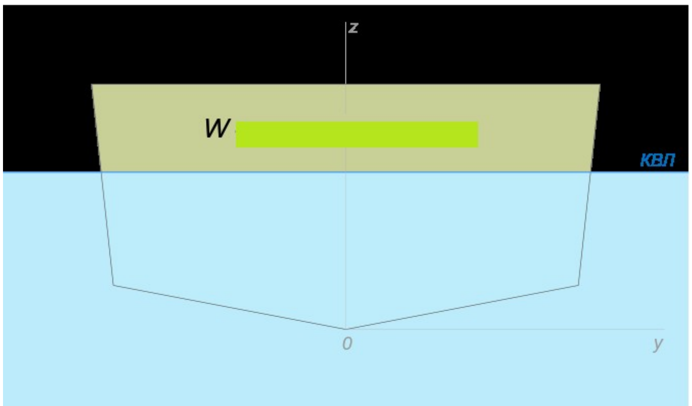


Fig. 4.9 Stock of buoyancy

$$P = \gamma\ (V_O - V_H),$$

$$P = \gamma\ V,$$

$$\rho\ F_z = dp\ /dz$$

$$\rho\ F_z\ dz = dp$$

$$p = p_0 + \rho\ g\ H,$$

$$p = \rho \cdot g \cdot h$$

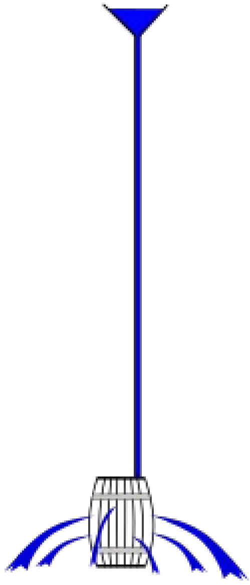


Fig. 4.10 The Pascal Experience

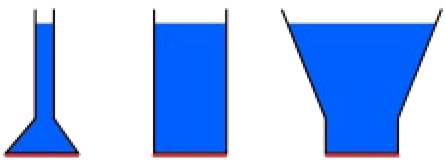


Fig. 4.11 The pressure at the bottom of all three vessels is the same

$$F_1 = (p_0 + \rho g h_1)\, A$$

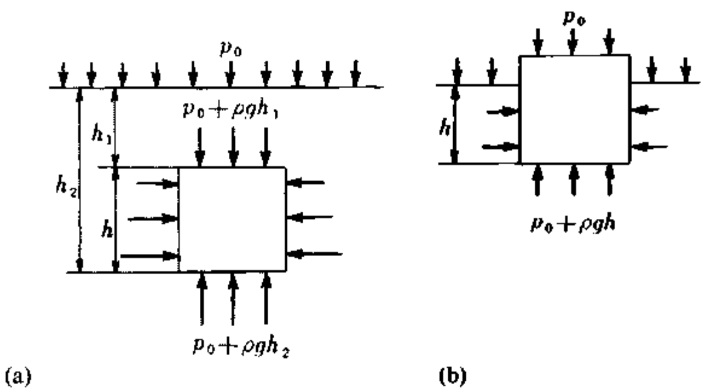


Fig. 4.12 Cube in liquid

$$F_2 = (p_0 + \rho g h_2)A$$

$$F = F_2\text{-}F_1 = pg(h_2 - h_1)A = \rho g h A = \rho g V$$

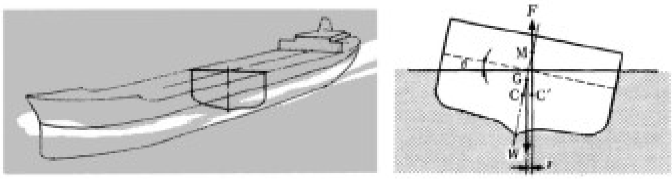


Fig. 4.13 Stability of a ship

Control questions

1. What is the fluid pressure at rest?
2. How does the pressure at the fluid change at rest depending on the depth?
3. What are the properties of hydrostatic pressure?
4. What is the basic equation of hydrostatics?
5. What is it the law of Archimedes?
6. Describe the equilibrium of a body floating on the surface of a liquid.
7. What is it the behavior of a body partially immersed in a liquid?
8. What is it the buoyancy of a ship (vessel)?
9. What is it buoyancy?
10. Describe a liquid in a homogeneous field of mass forces.
11. Describe Pascal's paradox.
12. Why is the ship sailing?

Lecture 5. Flow in pipes

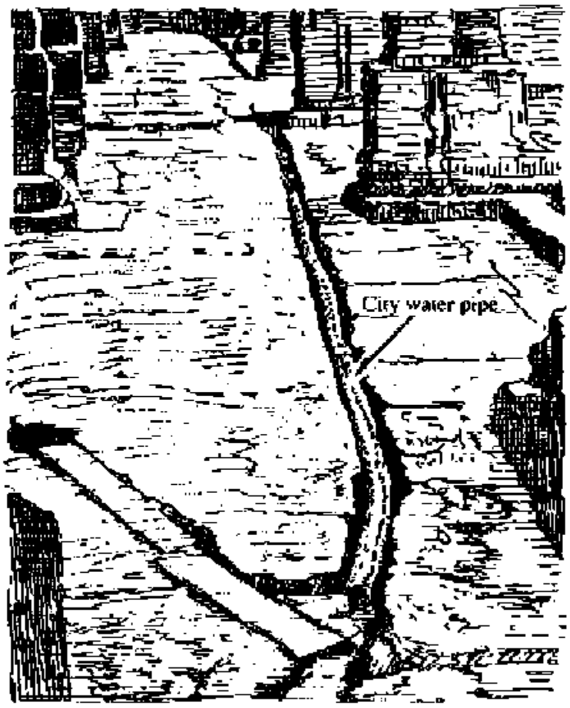


Fig. 5.1 Lead city water pipe (Roman remains, Bath, England)

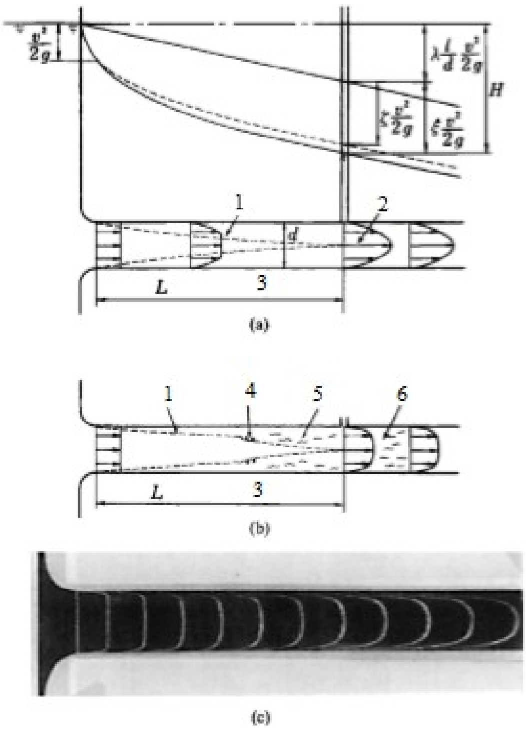


Fig. 5.2 Flow in a circular pipe: (a) laminar flow; (b) turbulent flow; (c) laminar flow (flow visualisation using hydrogen bubble method):
1 – laminar boundary layer; 2 - Laminar flow; 3 – inlet region; 4 – transition flow; 5 –

turbulent boundary layer; 6 - turbulent flow

Laminar flow:
L = 0.065 *Red* computation by Boussinesq experiment by Nikuradse
L = 0.06 *Red* computation by Asao, Iwanami and Mori
Turbulent flow:
L = 0.693 $Re^{1/4}d$ computation by Latzko
L = (25 ~ 40)*d* experiment by Nikuradse

$H = \lambda\, l/d\; v^2/2g + \zeta\, v^2/2g$ (5.1)

$$E = \int_0^{l/2} 2\pi r u \frac{\rho u^2}{2}\, dr \quad (5.2)$$

$E' = \pi d^2/4\; v\; \rho v^2/2$

$E/[1/4\; \pi\; d^2\, v\, \rho\, g] = \zeta\, v^2/2g$ (5.3)

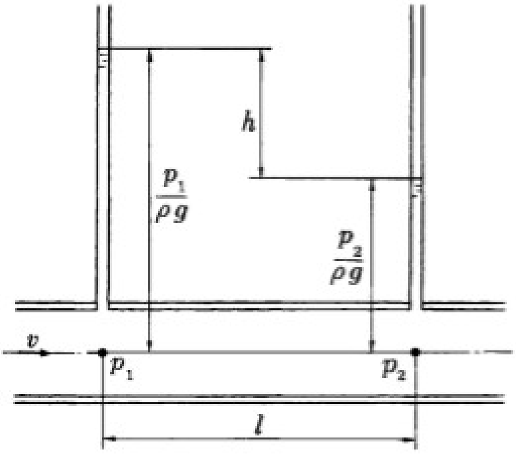


Fig. 5.3 Pipe frictional loss

$$h = \lambda\, l/d\; v^2/2g \quad (5.4)$$

$\lambda = 64\; \mu/\rho v d = 64/Re$ (5.5)

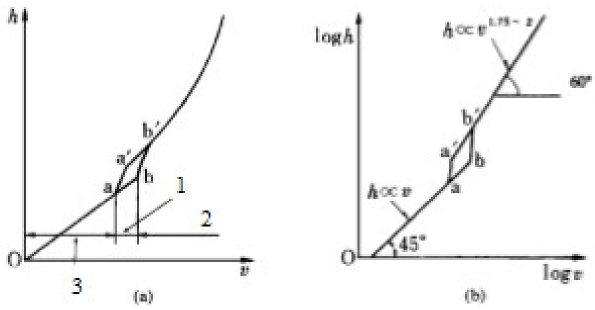


Figure 5.4 Relationship between flow velocity and loss head:
1 – transitional region; 2- turbulent flow; 3 - Laminar flow

$\varepsilon \leq 5\,\upsilon/v$ (5.6)

$\lambda = 0.3164Re^{-1/4}$ *(Re = $3 \times 10^3 \sim 1 \times 10^5$)* (5.7)

$\lambda = 0.0032 + 0.221\ Re^{-0.237}$ *(Re = $10^5 \sim 3 \times 10^6$)* (5.8)

$\lambda = 1/[2\log_{10}(Re\sqrt{\lambda}) - 0.8]^2$ *(Re = $3 \times 10^3 \sim 3 \times 10^6$)* (5.9)

$\lambda = 0.314/[0.7 - 1.65\log_{10}(Re) + (\log_{10} Re)^2]$(5.10)

$\varepsilon \leq 70\upsilon/v_*$ (5.11)

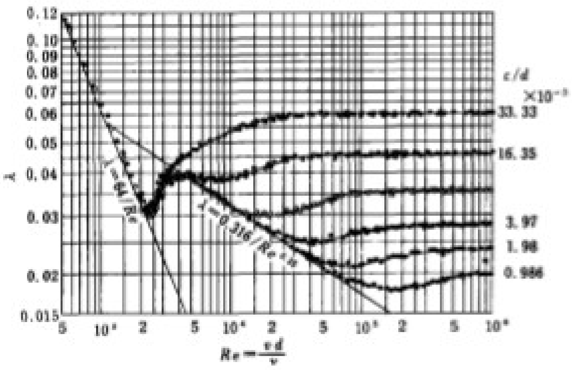


Figure 5.5 Friction coefficient of coarse circular pipe with sand drains

$\lambda = 1/[1.74 - 21og_{10}(2\varepsilon/d)]^2$ (5.12)

$u/v_* = 8.48 + 5.75\log_{10}(y/\varepsilon)$ (5.13)

$pghA = \tau_0 sl$ (5.14)

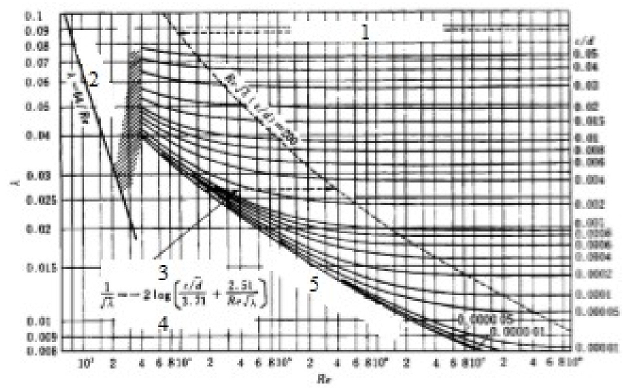


Fig. 5.6 Moody diagram:

1 – Rough pipe (Eqn. (5.12)); 2 – laminar flow; 3 – transition region; 4 – equation of Colebrook; 5 – smooth flow (Eqn. (5.9))

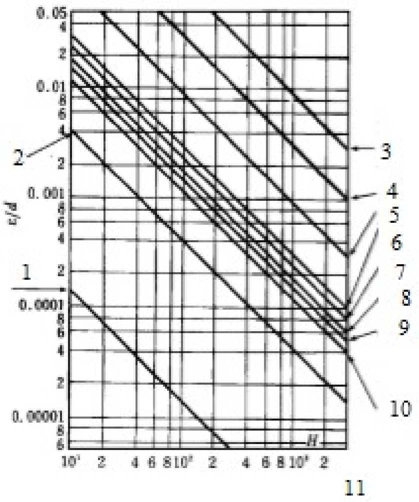


Fig. 5.7 Roughness of commercial pipe:
1 - Steel pipe (commercial) ε = 0.05; 2 - Seamless pipe ε = 0.0015; 3 - Rivet-jointed steel pipe ε - 9.15 mm; 4 - ε = 3.05; 5 - ε = 0.92; 6 - Concrete pipe ε=0.31; 7 - Cast-iron pipe ε = 0.26; 8 - Wooden pipe ε=0.92; 9 - Asphalt-coated cast iron pipe ε = 0.12 ε = 0.18; 10 - Zinc-galvanized iron pipe ε = 0.1; 11 - Pipe diameter, *d* (mm)

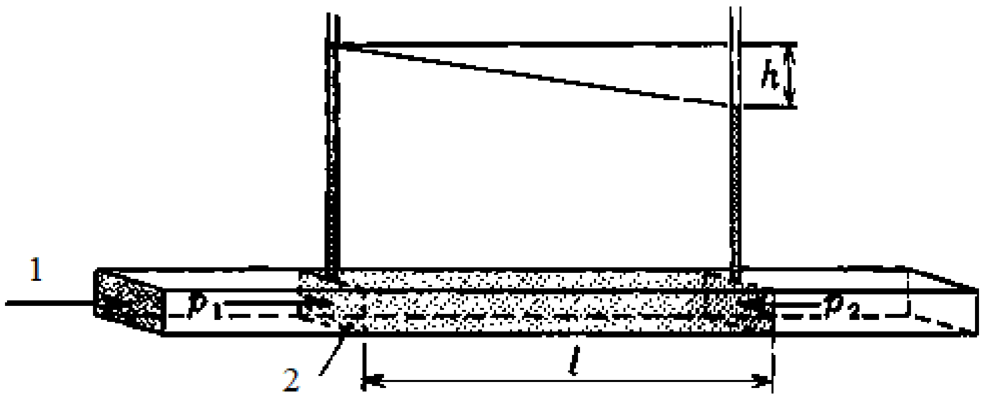


Figure 5.8 Flow in oblong pipe:
1 – Flow direction; 2 – Section area A

$H = \lambda\ l/4m\ v^2/2g \quad \lambda = f(Re, \varepsilon/4m)$ (5.15)

$4\ ab/2(a+b) = 2ab/(a+b) \quad 4\ (n/4)(d^2{}_2 — d^2{}_1)/\pi(d_1 — d_2) = d_2 — d_1$ (5.16)

$h_s = \zeta\ v^2/2g$ (5.17)

$h_s = (v_1 — v_2)^2\ /2g = (1 – A_1/A_2)^2\ v_1^2/2g$ (5.18)

$\boldsymbol{h_s} = \xi (v_1 — v_2)^2\ /2g$ (5.19)

$\boldsymbol{h_s} = \xi v_1^2/2g$ (5.20)

$\zeta = \xi\ (1 – A_1/A_2)^2$ (5.21)

$h_s = \xi v_1^2/2g$ (5.22)

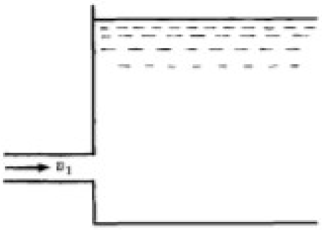


Figure 5.9 Outlet of pipe line

$$h_s = \frac{(v_c - v_2)}{2g} = \left(\frac{A_2}{A_c} - 1\right)^2 \frac{v_2^2}{2g} = \left(\frac{1}{C_c} - 1\right)^2 \frac{v_2^2}{2g}$$

$h_s = \zeta\, v^2/2g$ (5.24)

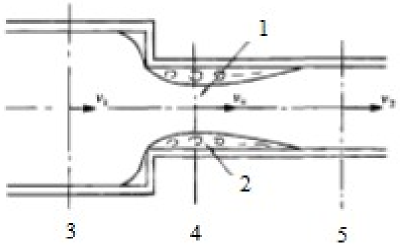


Fig. 5.10 Sudden contraction pipe:
1 – contraction; 2 – separation region; 3 – section 1, A_1; 4 – section 1, A_c; 5 – section 1, A_2

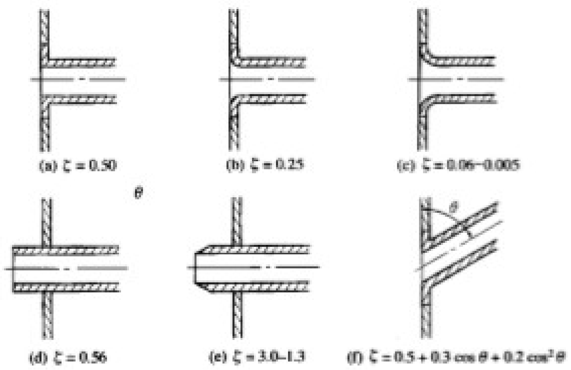


Fig. 5.11 Inlet shape and loss factor

$$Q = C\frac{\pi d^2}{4}\sqrt{\frac{2\Delta p}{\rho}} \quad (5.25)$$

$$C = \frac{1}{1.16 + 6.25\sigma^{-4.61}} \quad (5.26)$$

$$C = \frac{1}{1 + 5.3/\sqrt{\sigma}} \quad (5.27)$$

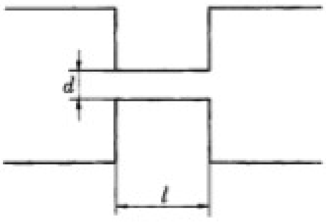


Figure 5.12 Choke

$h_s = \zeta(v_1 - v_2)^2/2g$ (5.28)

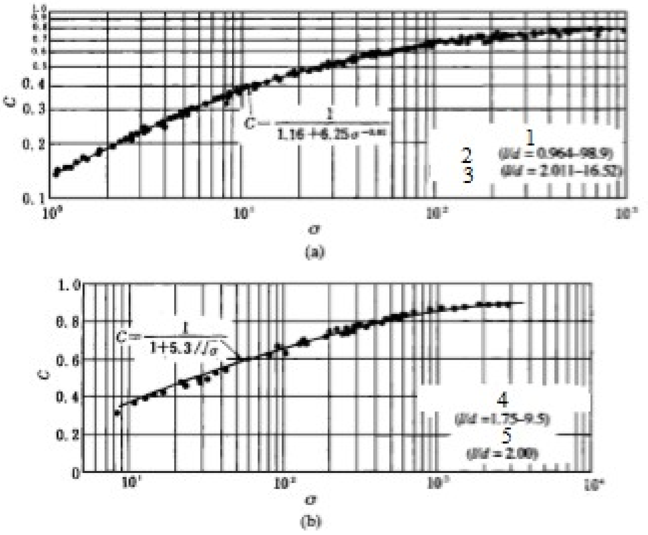


Fig. 5.13 Coefficient of discharge for cylindrical chokes: (a) entrance rounded; (b) entrance not rounded:
1 – experimental values; Chikawa, 2 – Iwanami; 3 – Nakayama; 4 – Hibi, Ichikawa, Miyagama, 3 – Nakayama

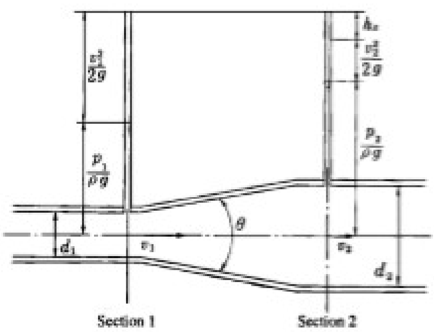


Fig. 5.14 Divergent flow

$p_1/\rho g + v_1^2/2g = p_2/\rho g + v_2^2/2g + h_s$

$(p_2 - p_1)/\rho g = (v_1^2 - v_2^2)/2g - h_s$ (5.29)

$(p_{2th} - p_1)/\rho g = (v_1^2 - v_2^2)/2g$ (5.30)

$\eta = (p_2 - p_1)/(p_{2th} - p_1) = 1 - h_s/[(v_1^2 - v_2^2)/2g]$ (5.31)

$\eta = 1 - \zeta(v_1 - v_2)/(v_1 + v_2) = 1 - \zeta(1 - A_1/A_2)/(1 + A_1/A_2)$ (5.32)

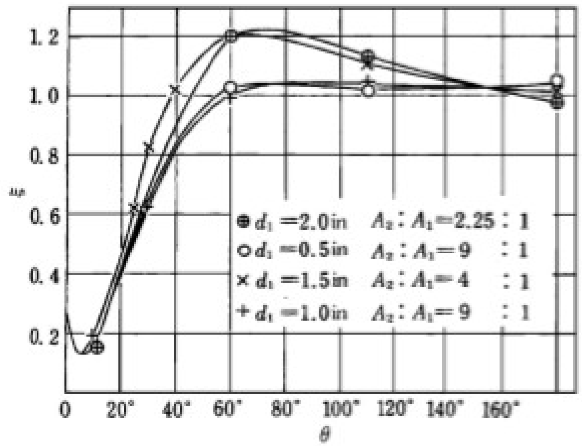


Fig. 5.15 Loss factor for divergent pipes

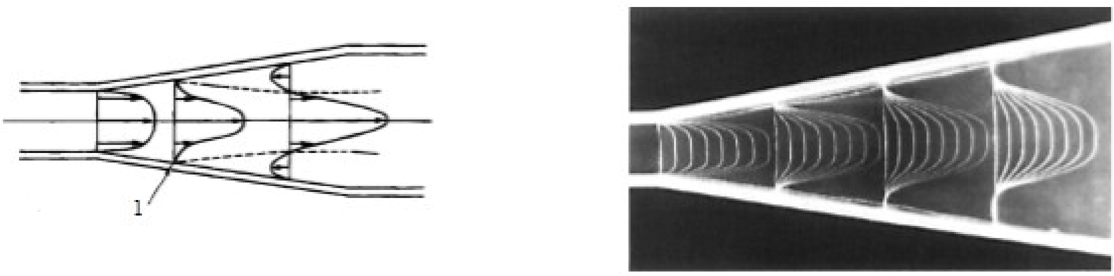


Fig. 5.16 Velocity distribution (a) and separation occurring (hydrogen bubble method, in water; inlet velocity 6 cm/s, *Re* (inlet port) = 90^0, divergent angle 20°) (b) in a divergent pipe:
1 - Separation point

$h_b = \zeta_b\, v^2/2g = (\zeta + \lambda\, l/d)\, v^2/2g$ (5.33)

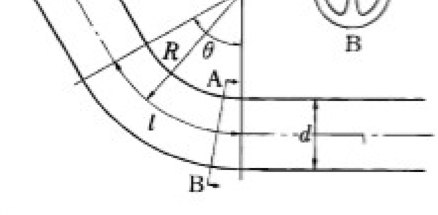


Fig. 5.18 Bend

$h_{s1} = \zeta_1 v_1^2/2g \quad h_{s2} = \zeta_2 v_1^2/2g \quad (5.34)$

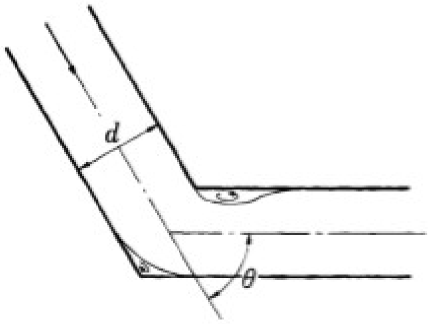


Figure 5.19 Elbow

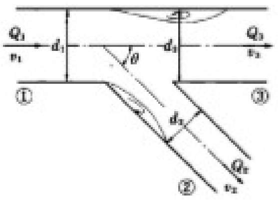


Figure 5.19 Pipe branch

$h_{s1} = \zeta_1 v_3^2/2g \quad h_{s2} = \zeta_2 v_3^2/2g \quad (5.35)$

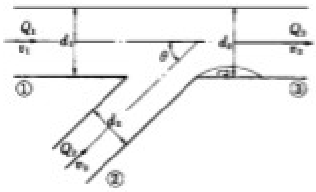


Fig. 5.20 Pipe junction

$\zeta = \boldsymbol{t/d}$ (5.36)

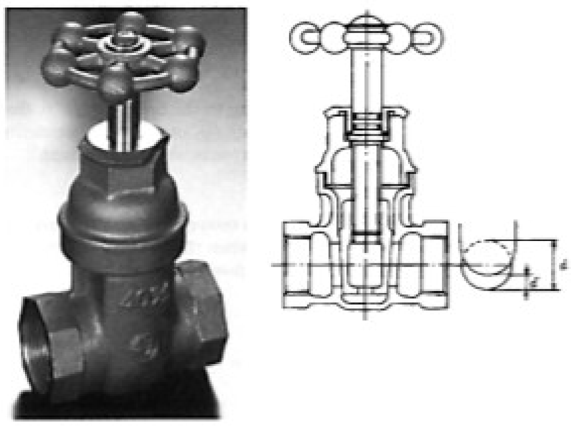


Fig. 5.21 Gate valve

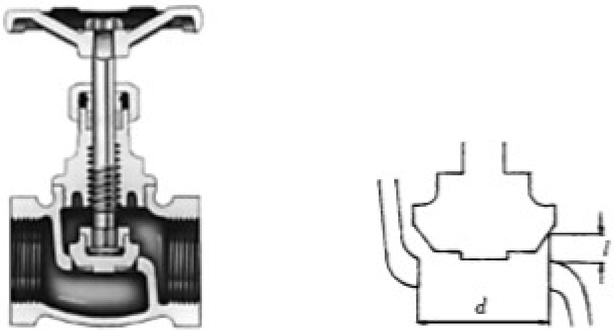


Fig. 5.22 Globe valve

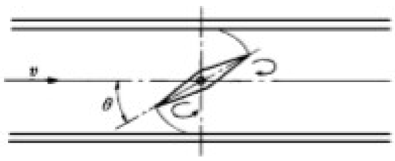


Fig. 5.23 Butterfly valve

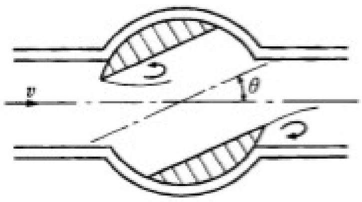
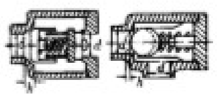
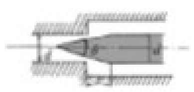
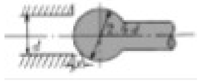
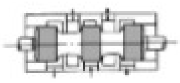


Fig. 5.24 Cock

Table 5.1 Loss factor for various valves

<table>
<tr><td>Valve</td><td>Loss coefficients, ζ</td></tr>
<tr><td>Relief valve
</td><td><table><tr><td>h/d</td><td>0.05</td><td>0.1</td><td>0.15</td><td>0.2</td><td>0.25</td><td>0.3</td></tr><tr><td>ζ</td><td>3.35</td><td>2.85</td><td>2.4</td><td>2.4</td><td>1.7</td><td>1.35</td></tr></table></td></tr>
<tr><td>Disc valve
</td><td>Throttle area $a = \pi dx$
Section area of valve seat hole $A = \pi d^2/4$
When x d/4 a = A
Loss coefficient $\zeta = 1.3 + 0.2(A/a)^2$</td></tr>
<tr><td>Needle valve
</td><td>$a=\pi(dx \tan\theta/2-x^2 \tan\theta/2)$
Ax =0 when x=0
$\zeta = 0.5 + 0.15(A/a)^2$</td></tr>
<tr><td>Ball valve</td><td></td></tr>
</table>

	$a \cong 0.75\pi dx$ $\zeta = 0.5 + 0.15(A/a)^2$
Spool valve	At full open positions $\zeta = 3 \sim 5.5$
	

$$h = (\lambda\, l/d + \Sigma\zeta)\, v^2/2g \quad (5.37)$$

$$h = (\lambda\, l/d + \Sigma\zeta + 1)\, v^2/2g \quad (5.38)$$

$H = H_a + h$ (5.39)

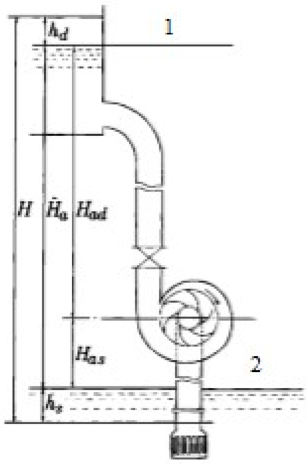


Fig. 5.25 Storage pump: *H* total head; H_a actual head; $H_{a,s}$ suction head; $H_{s,d}$ discharge head; h_s losses on suction s; h_d losses on discharge side:
1 – Discharge water level; 2 – Suction water level

$L_w = \rho g Q H$(5.40)

$L_w/L_S = \eta$ (5.41)

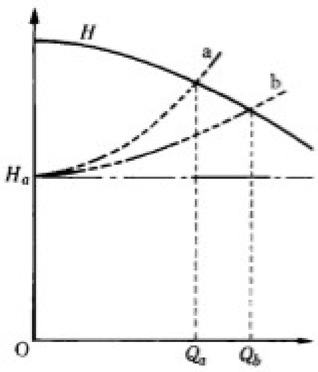


Fig. 5.26 Total head and load curve of pump:

a – valve opening small; b – valve opening large

Control questions

1. Describe the flow in the inlet region.
2. Describe the loss by pipe friction.
3. Describe the frictional loss on pipes other than circular pipes.
4. Describe the losses in pipe lines with sudden expansion and contraction of flow.
5. Describe the throttle in pipe lines.
6. Describe the divergent pipe and the divergent flow
7. Describe the loss whenever the flow direction changes (bend and elbow).
8. Describe pipe branch and pipe junction
9. Describe valves and taps.
10. Describe the total pipeline loss.
11. Describe pumping to higher levels.

Recommended literature

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Lecture 6. Fundamentals of flow

dx/u = dy/v   (6.1)

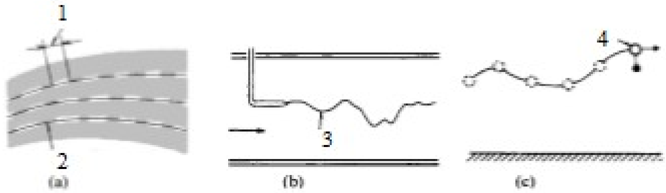

Fig. 6.1 Lines showing flows:
1 – Shutter opening time; 2 – Streamline; 3 – Dyes; 4 – Balloon
a) Streamline; b) Streak line; c) Path line

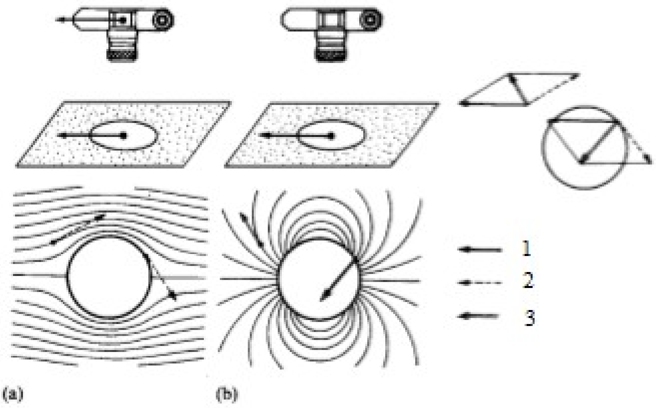

Fig. 6.2 Relative streamlines and absolute streamlines:
1 - Movement of body; 2 - Relative velocity; 3 - Absolute velocity
(a) Relative streamlines (b) Absolute streamlines

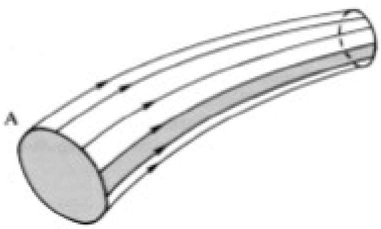

Fig. 6.3 Stream tube

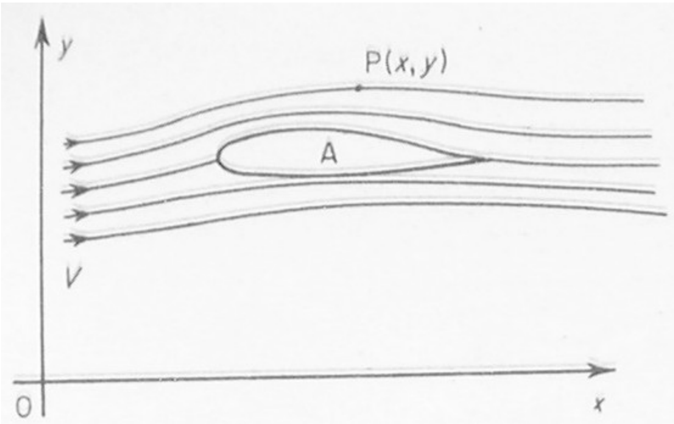

Fig. 6.4 (a) Air moves at speed *V* past axes fixed relative to aerofoil

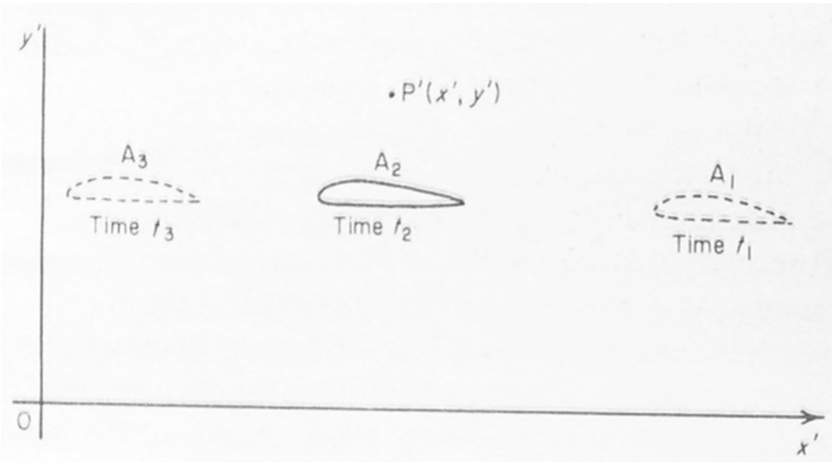

Fig. 6.4 (b) Aerofoil moves at speed *V* through air initially at rest. Axes Ox' Oy' fixed relative to undisturbed air.

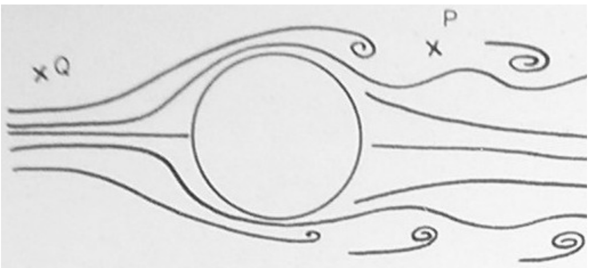

Fig. 6.5 True unsteady flow

u — $u(x, y, z,t)$ $v = v(x, y, z,t)$ $w = w(x, y, z, t)$ (6.2)

u — $u(x, y,t)$ v — $v(x, y, t)$ (6.3)

$u = u(x, t)$ (6.4)

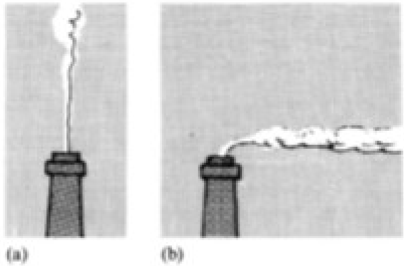

Figure 6.6 Smoke from a chimney

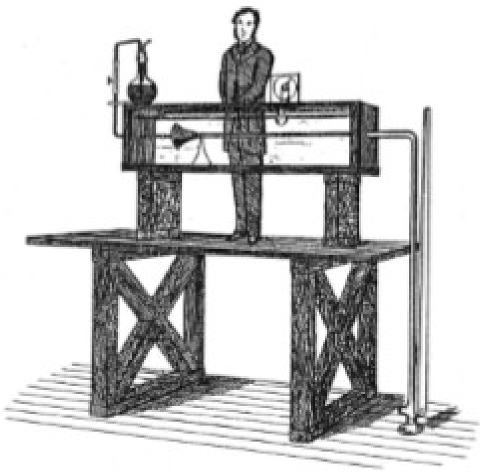

Figure 6.7 Reynolds' experiment

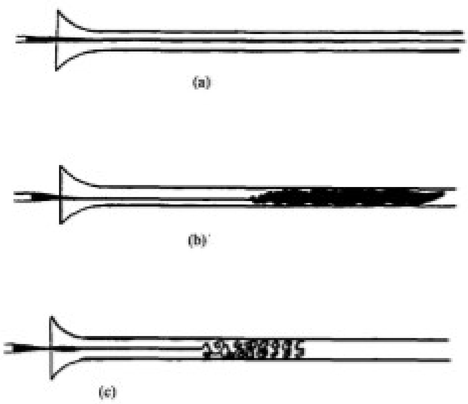

Figure 6.8 Reynolds' sketch on transition from laminar flow to turbulent flow:
a) Laminar flow; b) Turbulent flow; c) Turbulent flow (observed by electric spark)

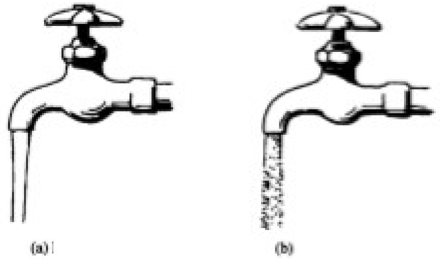

Fig. 6.9 Water flowing from a faucet:
a) Laminar flow; b) Turbulent flow

$Re = \rho vd/\mu = vd/\upsilon$ (6.5)

Fig. 6.10 Osborne Reynolds (1842-1912)
Mathematician and physicist of Manchester, England. His research covered all the fields of physio and engineering - mechanics, thermodynamics, electricity, navigation, rolling friction and steam engine performance. He was the first to clarify the phenomenon of cavitation and the accompanying noise. He discovered the difference between laminar and turbulent flows and the dimensionless number, the Reynolds number, which characterises these flows. His lasting contribution was the derivation of the momentum equation of viscous fluid for turbulent flow and the theory of oil-film lubrication.

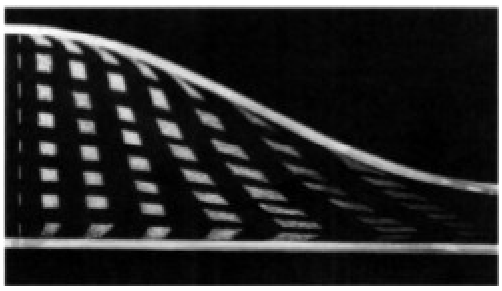

Figure 6.11 Deformation and rotation of fluid particles through a narrowing channel

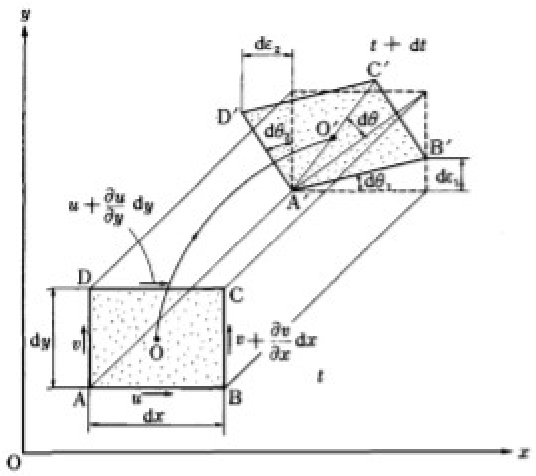

Figure 6.12 Deformation of elementary rectangle of fluid

$d\varepsilon_1 = \partial v/\partial x \; dx \; dt \qquad d\varepsilon_2 = \partial u/\partial y \; dy \; dt$
$d\theta_1 = d\varepsilon_1/dx = \partial v/\partial x \; dt \qquad d\theta_2 = d\varepsilon_2/dy = -\partial u/\partial y \; dt$

$\omega_1 = d\theta_1/dt = \partial v/\partial x \quad \omega_2 = d\theta_2/dt = -\partial u/\partial y$

$\omega = ½ (\omega_1 + \omega_2) = ½ (\partial v/\partial x - \partial u/\partial y) \quad (6.6)$

$\zeta = \partial v/\partial x - \partial u/\partial y$

$\partial v/\partial x - \partial u/\partial y = 0 \quad (6.8)$

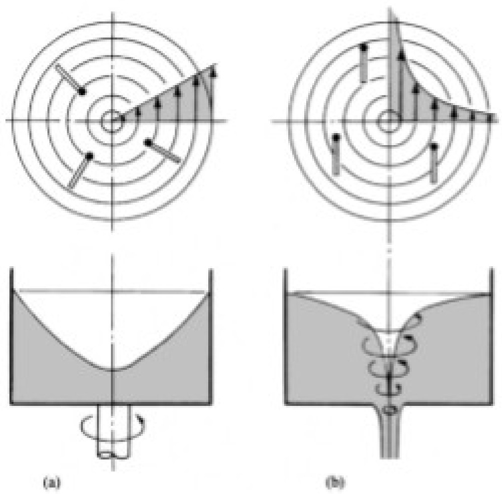

Fig. 6.13 Vortex flow:
a) Forced vortex flow; b) Free vortex flow

$\zeta = \text{rot } V = \text{curl } V = [\partial w/\partial y - \partial v/\partial z, \partial u/\partial z - \partial w/\partial x, \partial v/\partial x - \partial u/\partial y]$

$i \; \partial/\partial x + j \; \partial/\partial y + k \; \partial/\partial z$

Fig. 6.14 Tornado

$$\Gamma = \oint v_s^{'} ds = \oint v_s cos\theta ds$$

$$d\Gamma = udx + \left(v + \frac{\partial v}{\partial x}dx\right)dy - \left(u + \frac{\partial u}{\partial y}dy\right)dx - vdy = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy = \zeta dxdy = \zeta dA$$

$$\Gamma = \oint v_s^{'} ds = \oint \zeta dA$$

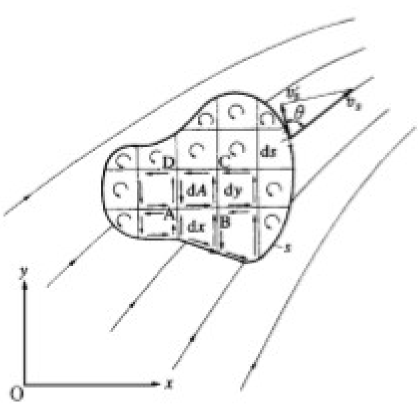

Fig. 6.15 Circulation

Fig. 6.16 George Gabriel Stokes (1819-1903)
Mathematician and physicist. He was born in Sligo in Ireland, received his education at Cambridge, became the professor of mathematics and remained in England for the rest of his life as a theoretical physicist. More than 100 of his papers were presented to the Royal Society, and ranged over many fields, including in particular that of hydrodynamics. His 1845 paper includes the derivation of the Navier-Stokes equations.

Control questions

1. Describe two methods for studying the movement of flow
2. Describe streamline and stream tube
3. Describe steady flow and unsteady flow
4. Describe true unsteady flow.
5. Describe three concepts which are useful in describing fluid flows.
6. Describe three-dimensional, two-dimensional and one-dimensional flows.
7. Describe laminar flow and turbulent flow
8. Describe the Reynolds number.
9. Describe incompressible and compressible liquids.
10. Describe rotation and spinning of a liquid
11. Describe the circulation

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11. Falkovich G. Fluid Mechanics. Cambridge University Press, The Edinburgh Building, Cambridge CB2 2RU, UK, 2011. - 180 p.
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Lecture 7. One-dimensional flow: mechanism for conservation of flow properties

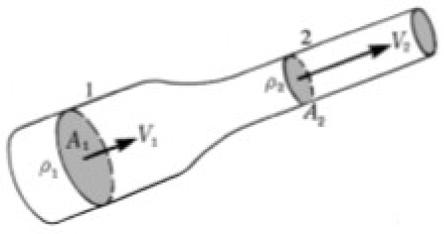

Fig. 7.1 Mass flow rate passing through any section is constant

$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

$\rho A v$ = constant (7.1)

Av = constant (7.2)

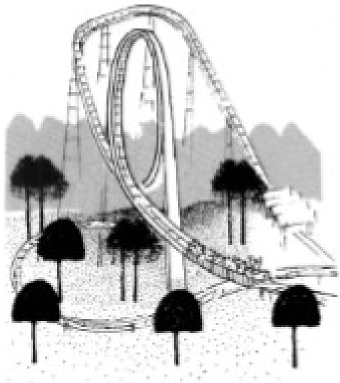

Fig. 7.2 Movement of roller-coaster

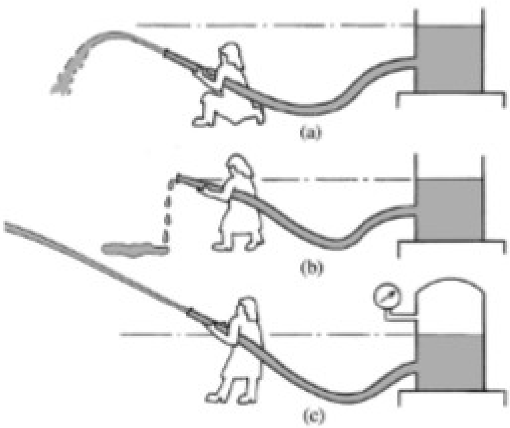

Fig. 7.3 Conservation of fluid energy

$\rho dAds\ dv/dt = -dA\ \partial p/\partial s\ ds — \rho g dAds\cos\theta$

$dv/dt = -1/\rho\ \partial p/\partial s\ ds — g\cos\theta$ (7.3)

$dv = \partial v/\partial t\ dt + \partial v/\partial s\ ds$

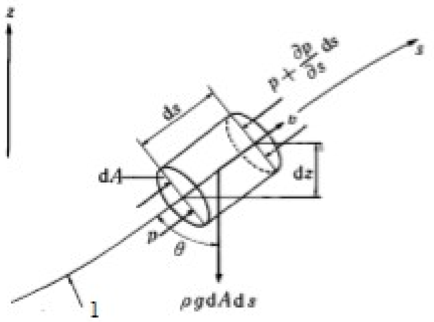

Fig. 7.4 Force acting on fluid on streamline

$dv/dt = \partial v/\partial t + \partial v/\partial t\ ds/dt = \partial v/\partial t + v\partial v/\partial s$

$\cos\theta = dz/ds$

$\partial v/\partial t + v\partial v/\partial s = -1/\rho\ \partial p/\partial s + g\ dz/ds$ (7.4)

$v\ dv/ds = 1/\rho\ dp/ds - g\ dz/ds$ (7. 5)

$v^2/2 + \int dp/\rho + gz = \text{constant}$ (7.6)

$v^2/2 + p/\rho + gz = \text{constant}$ (7.7)

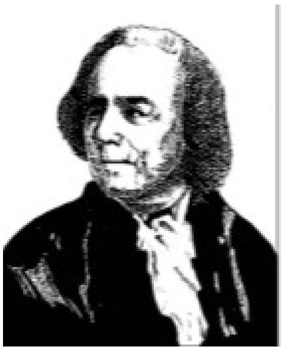

Fig. 7.5 Leonhard Euler (1707-83)
Mathematician born near Basle in Switzerland. A pupil of Johann Bernoulli and a dose friend of Daniel Bernoulli. Contributed enormously to the mathematical development of Newtonian mechanics, while formulating the equations of motion of a perfect fluid and solid. Lost his sight in one eye and then both eyes, as a result of a disease, but still continued his research.

$v^2/2g + p/\rho g + z = H = \text{constant}$ (7.8)

$\rho v^2/2 + p/g + \rho gz = \text{constant}$ (7.9)

Fig. 7.6 Daniel Bernoulli (1700-82)
Mathematician born in Groningen in the Netherlands. A good friend of Euler. Made efforts to popularise the law of fluid motion, while tackling various novel problems in fluid statics and dynamics. Originated the Latin word hydrodynamica, meaning fluid dynamic.

$\rho v^2/2 + p_s = p_t$ (7.10)

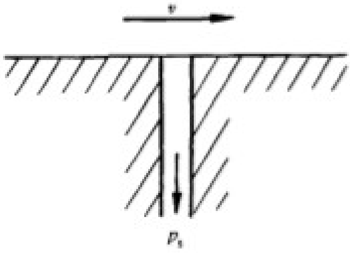

Fig. 7.7 Picking out of static pressure

$v_1^2/2g + p_1/\rho g = v_2^2/2g + p_2/\rho g$ (7.11)

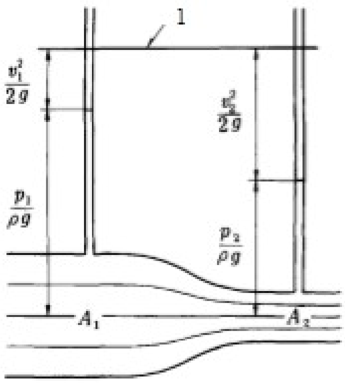

Fig. 7.8 Exchange between pressure head and velocity head:
1 – Total head

$v_1 A_1 = v_2 A_2$ (7.12)

$v_1^2/2 + p_1/\rho + z_1 = v_2^2/2 + p_2/\rho + z_2 + h_2 = v_3^2/2 + p_3/\rho + z_3 + h_3$ (7.13)

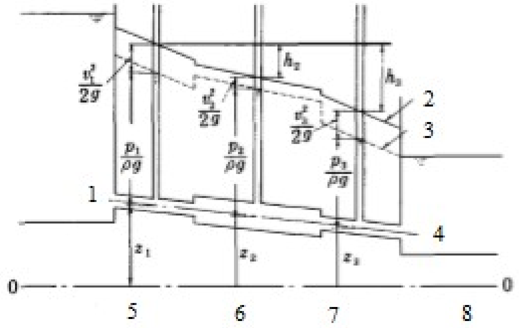

Fig. 7.9 Hydraulic grade line and energy line:
1 – Water tank 1; 2 - energy line; 3 - Hydraulic grade line; 4 - Water tank 2; 5 – Section 1

$p_1/\rho g + v_1^2/2g + z_1 = p_2/\rho g + v_2^2/2g + z_2$

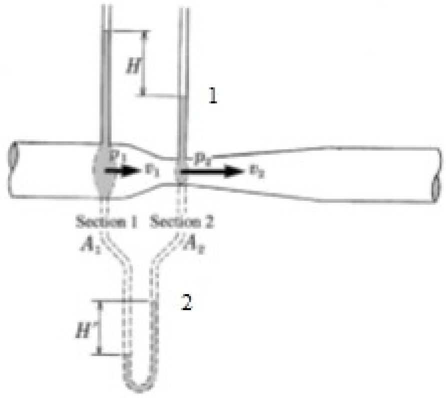

Figure 7.10 Venturi tube:
1 – In the case of water; 1 – In the case of air

Figure 7.11 Giovanni Battista Venturi (1746-1822)
Italian physicist. After experiencing life as a priest, teacher and auditor, finally became a professor of experimental physio. Studied the effects of eddies and the flow rates at various forms of mouthpieces fitted to an orifice, and clarified the basic principles of the Venturi tube and the hydraulic jump in an open water channel.

$z_1 = z_2$
$(v_2^2 - v_1^2)/2g = (p_1 - p_2)/\rho g$
$v_1 = v_2 A_2/A_1$

$$v_2 = \frac{1}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2g \frac{p_1 - p_2}{\rho g}} \quad (7.14)$$

$(p_1 - p_2)/\rho g = H$

$$Q = A_2 v_2 = \frac{A_2}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2gH} \quad (7.15)$$

$$Q = C \frac{A_2}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2gH} \quad (7.16)$$

Figure 7.12 Henry de Pitot (1695-1771)
Born in Aramon in France. Studied mathematics and physics in Paris. As a civil engineer, undertook the drainage of marshy lands, construction of bridges and city water systems, and flood countermeasures. His books cover structures, land survey, astronomy, mathematics, sanitary equipment and theoretical ship steering in addition to hydraulics. The famous Pitot tube was announced in 1732 as a device to measure flow velocity.

Fig. 7.13 Pitot's first experiment:
As expected, whenever the tube faces into a flow, water in tube goes up. From its height, the flow velocity can be computed.

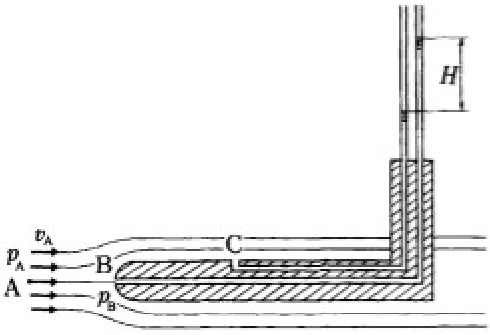

Fig. 7.14 Pitot tube

$p_A/\rho g + v^2_A/2g = p_B/\rho g$

$$v_A = \sqrt{2g\frac{p_B - p_C}{\rho}} \quad (7.17)$$

$$v_A = \sqrt{2\frac{p_B - p_C}{\rho}}$$

$$v_A = \sqrt{2gH}$$

: $$v_A = C_v\sqrt{2gH}$$

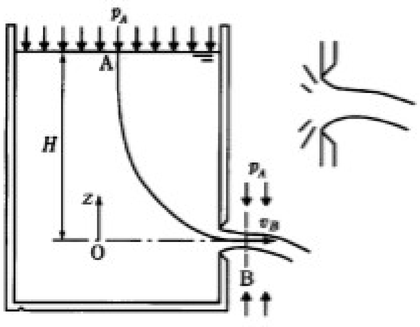

Fig. 7.15 Flow through a small hole (1)

$p_A/\rho g + v^2_A/2g + z_A = p_A/\rho g + v^2_B/2g + z_B$

$p_A/\rho g + H = p_A/\rho g + v^2_B/2g$

(7.21) $$v_A = \sqrt{2gH}$$

$a_c = C_c\, a$(7.22)

$v = C_v\, v_B = C_v \sqrt{(2gH)}$ (7.23)

$Q = C_c\, a\, C_v\, v_B = C_c\, C_v \sqrt{(2gH)}$ (7.24)

$Q = C\, a \sqrt{(2gH)}$ (7.25)

$v = \sqrt{(2gH)}$

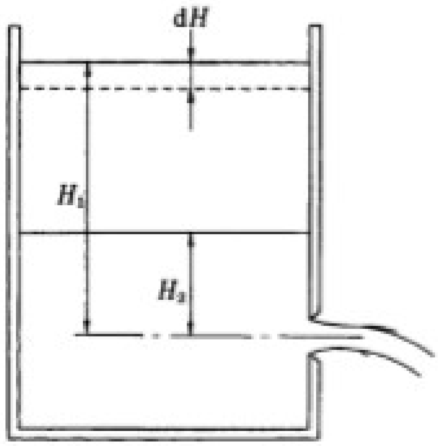

Fig. 7.12 Flow through a small hole (2)

$$dQ = Ca\sqrt{2gH}\,dt = -dHA$$

$$dt = \frac{-AdH}{Ca\sqrt{2gH}}$$

$$\int_{t_1}^{t_2} dt = \frac{-A}{Ca\sqrt{2g}} \int_{u_1}^{u_2} \frac{dH}{\sqrt{H}}$$

$$t_2 - t_1 = \frac{2A}{Ca\sqrt{2g}}\left(\sqrt{H_1} - \sqrt{H_2}\right)$$

$dQ = C\, a \sqrt{(2gH)}\, dt = -dH\, A = -dH\, \pi r^2$

$v = -\,dH/dt = [C\,a\,\sqrt{(2gH)}]/\pi r^2$ (7.27)

(7.28)

$$H = \left(\frac{\pi v}{C a \sqrt{2g}} \right)^2 r^4$$

$H \propto r^4$

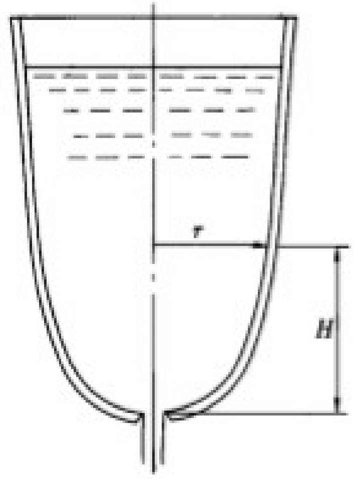

Fig. 7.13 Flow through a small hole (3)

Fig. 7.14 Egyptian water clock 3400 years old (London Science Museum)

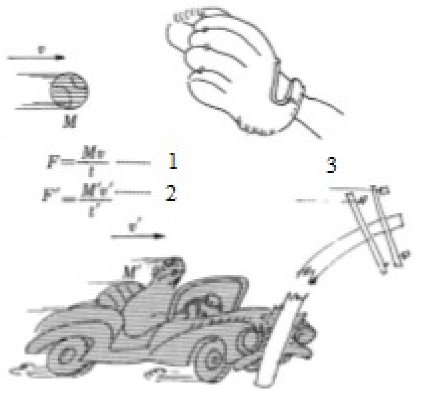

Fig. 7.16 Car does not stop immediately:
1 – small; 2 – large; 3 – Where t and t' are the respective times from collision to stopping

$F = (M_{v2} - M_{v1})/t$ (7.31)

$$\left.\begin{aligned} -F_x + A_1p_1\cos\alpha_1 - A_2p_2\cos\alpha_2 = m(v_2\cos\alpha_2 - v_1\cos\alpha_1) \\ -F_y + A_1p_1\sin\alpha_1 - A_2p_2\sin\alpha_2 = m(v_2\sin\alpha_2 - v_1\sin\alpha_1) \end{aligned}\right\} \quad (7.32)$$

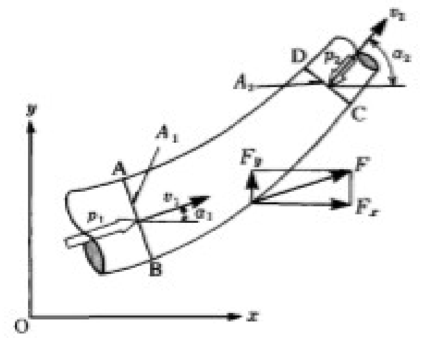

Fig. 7.17 Flow in a curved pipe

$m = \rho Q = \rho A_1 v_1 = \rho A_2 v_2 = \rho Q$

$F_x = m(v_1\cos\alpha_1 - v_2\cos\alpha_2) + A_1 p_1 \cos\alpha_1 - A_2 p_2 \cos\alpha_2$

$F_y = m(v_1\sin\alpha_1 - v_2\sin\alpha_2) + A_1 p_1 \sin\alpha_1 - A_2 p_2 \sin\alpha_2$

$F_x = \sqrt{F_x^2 - F_y^2}$ (7.34)

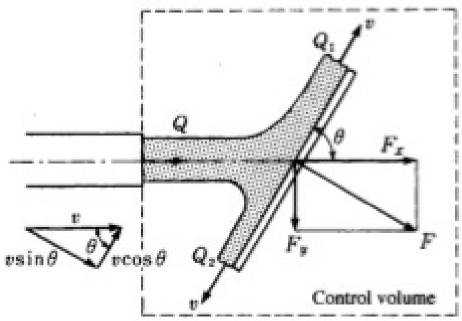

Fig. 7.18 Force of jet acting on a flat plate at rest

$F = \rho Qv \sin\theta$ (7.35)

$F_x = F \sin\theta = \rho Qv \sin^2\theta$ (7.36)

$F_y y = F \cos\theta = pQv \sin\theta \cos\theta$ (7.37)

$Q_1 = Q (1 + \cos\theta) / 2$ (7.38)
$Q_2 = Q (l - \cos\theta) / 2$ (7.39)

$F = pQ'(v - u) \sin\theta = pQ (u - v)^2/v \sin\theta$ (7.40)

$= Mr^2 v/r = Mrv$ (7.48)

torque (rotational couple) on this body is given by
Torque = change of angular momentum
= moment of inertia x angular acceleration (7.49)

Fig. 7.19 Ice skater
(a) Slow spin(b) Quick spin

$$T + A_2 p_2 r_2 \cos\alpha_2 - A_1 p_1 r_1 \cos\alpha_1 = m(r_2 v_2 \cos\alpha_2 - r_1 v_1 \cos\alpha_1) \quad (7.50)$$

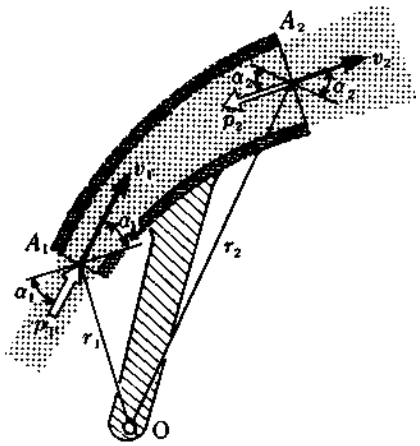

Fig. 7.20 Flow in curved tube supported so as to turn around shaft θ

Control questions
1. What is the essence of both the law of conservation of mass and continuity equation?
2. What is the essence of the law of conservation of energy?
3. What is the line of free surface and the line of total flow energy?
4. What idea is embedded in the Pitot tube and how is it used to determine the flow velocity?
5. What are the applications of the Bernoulli equation?
6. What is the essence of the law of conservation of motion?
7. What is the essence of conservation of angular momentum?

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Lecture 8 Two-dimensional flow

$$\frac{\delta s}{\delta t} = \frac{ds}{dt} = q$$

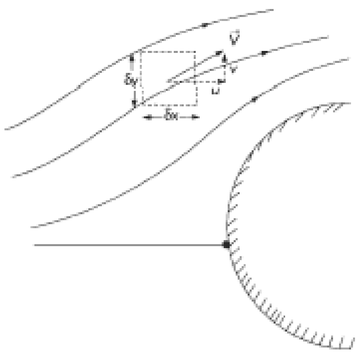

Fig. 8.1 An infinitesimal control volume in a typical two-dimensional flow field

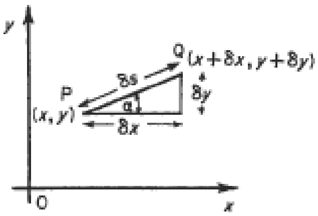

Fig. 8.2 Particle move in a Cartesian coordinate system

$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$

$q^2=u^2+v^2$

$(\delta s)^2 = (\delta r)^2 + (r\delta\theta)^2$

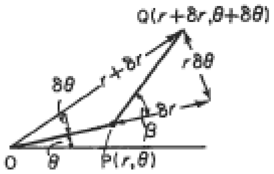

Fig. 8.3 Particle move in a polar coordinate system

$q^2 = q^2_n + q^2_t$

$\beta = \tan^{-1} q_t/q_n$

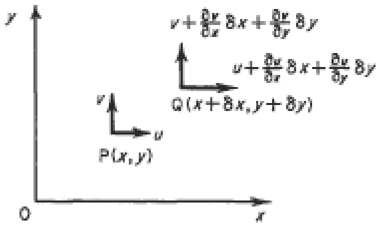

Fig. 8.4 Move of fluid particle in a two-dimensional flow

$u + \delta u = u + \partial u/\partial x\ \delta x + \partial u/\partial y\ \delta y$ and $v + \delta v = v + \partial v/\partial y\ \delta y$

$d(u+\delta u)/dt = \partial u/\partial t + \partial u/\partial x\ dx/dt + \partial u/\partial y\ dy/dt$

$= \partial u/\partial t + u\,\partial u/\partial x + v\,\partial u/\partial y$

$d(v+\delta v)/dt = u\,\partial v/\partial x + v\,\partial v/\partial y$

$p + \delta p = p + \partial p/\partial x\,\delta x + \partial p/\partial y\,\delta y$

$(\rho u - \partial(\rho u)/\partial x\,\delta x/2)\,\delta y \times 1$ (8.1)

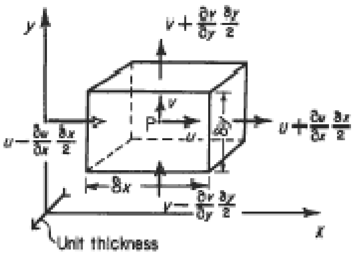

Fig. 8.5 Rectangular space of volume δx x δy x 1 at the point P (x, y) where the velocity components are u and v and the density is ρ

$(\rho u + \partial(\rho u)/\partial x\,\delta x/2)\,\delta y \times 1$ (8.2)

$-\partial(\rho u)/\partial x\,\delta x\,\delta y$

$-\partial(\rho v)/\partial y\,\delta x\,\delta y$

$-(\partial(\rho v)/\partial y + \partial(\rho v)/\partial y)\,\delta x\,\delta y$ (8.3)

$\partial(\rho \times \text{volume})/\partial t$

$\partial\rho/\partial t\,\delta x\,\delta y \times 1$ (8.4)

$\partial\rho/\partial t + \partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0$

$\partial\rho/\partial t + u\partial\rho/\partial x + v\partial\rho/\partial y + \rho(\partial u/\partial x + \partial v/\partial y) = 0$ (8.5)

$\partial u/\partial x + \partial v/\partial y = 0$ (8.6)

$\partial u/\partial x = -\partial v/\partial y$

$\partial\rho/\partial t + u\partial\rho/\partial x + v\partial\rho/\partial y + w\partial\rho/\partial z + \rho(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) = 0$

$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$

$(\rho q_n - \partial(\rho q_n)/\partial r\,\delta r/2)(r - \delta r/2)\,\delta\theta - (\rho q_n + \partial(\rho q_n)/\partial r\,\delta r/2)(r + \delta r/2)\,\delta\theta$
$= -\rho q_n\,\delta r\delta\theta - \partial(\rho q_n)/\partial r\;r\,\delta r\delta\theta$

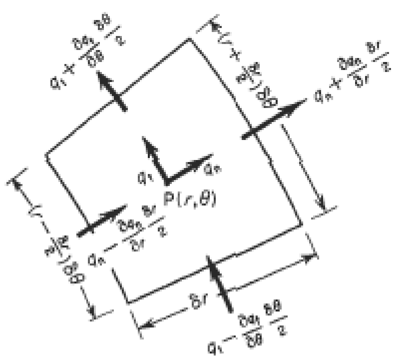

Fig. 8.6 Rectangular element at P (r, θ) in a system of polar coordinates

$(\rho q_t - \partial(\rho q_t)/\partial\theta\,\delta\theta/2)\,\delta r - (\rho q_t + \partial(\rho q_t)/\partial\theta\,\delta\theta/2)\,\delta r = -\partial(\rho q_t)/\partial\theta\,\delta r\delta\theta$

$= -(\rho q_n/r + \partial(\rho q_n)/\partial r + 1/r\ \partial(\rho q_t)/\partial\theta)\ \delta r\delta\theta \quad (8.7)$

$= \partial(\rho r\delta r\delta\theta)/\partial t \quad (8.8)$

$\rho q_n/r + \partial\rho/\partial t + \partial(\rho q_n)/\partial r + 1/r\ \partial(\rho q_t)/\partial\theta = 0$

$\partial(\rho\ r\ q_n)/\partial r + \partial(\rho q_t)/\partial\theta = 0$

$q_n/\partial r + \partial q_n/\partial r + 1/r\ \partial q_t/\partial\theta = 0$

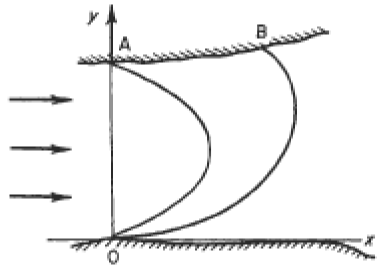

Fig. 8.7 Banks of a shallow river of a constant depth

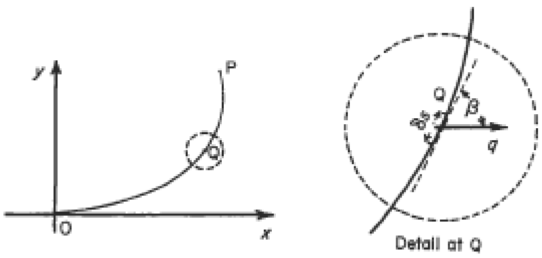

Fig. 8.8 A pair of coordinate axes set in a two-dimensional air stream

$$\int_{OP} q\sin\beta ds$$

$$\psi_p = \int_{OP} q\sin\beta ds$$

Fig. 8.9 Sign convention for stream functions

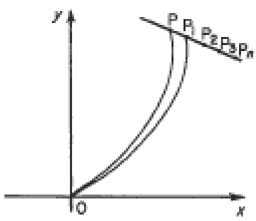

Fig. 8.10 The streamline

$\delta\psi = u\delta y - v\delta x$

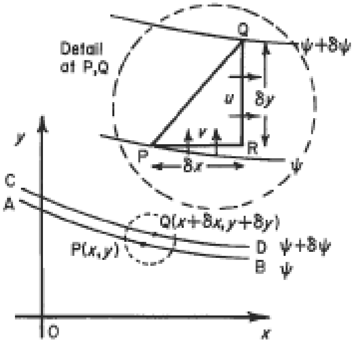

Fig. 8.11a Velocity components in terms of ψ. in Cartesian coordinates

$\delta\psi = \partial\psi/\partial x \; \delta x + \partial\psi/\partial y \; \delta y$

$u = \partial\psi/\partial y$

$v = -\partial\psi/\partial x$

$\delta\psi = -q_t\delta r + q_n(r + \delta r)\,\delta\theta = -q_t\delta r + q_n\, r\delta\theta + q_n\,\delta r\,\delta\theta$

$\delta\psi = -q_t\delta r + q_n r\,\delta\theta$ (8.9)

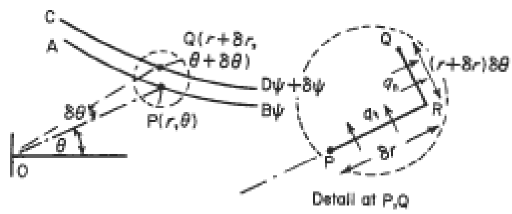

Fig. 8.11b Velocity components in terms of ψ in polar coordinates

$\delta\psi = \partial\psi/\partial r \; \delta r + \partial\psi/\partial\theta \; \delta\theta$ (8.10)

$q_t = -\,\partial\psi/\partial r$ (8.10a)
$q_n = 1/r\; \partial\psi/\partial\theta$ (8.10b)

$q = \partial\psi/\partial n$

Fig. 8.12 Velocity q in any direction

$(\sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j})\,\delta x \times 1$, $(\sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j})\delta y \times 1$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

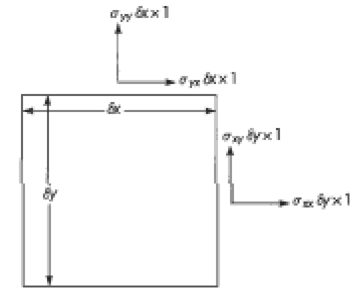

Fig. 8.13 A square infinitesimal control volume

$$\underbrace{Rate of increase of momentum within the c.v.}_{(i)}$$

$$+\underbrace{Net rate at which momentum leaves the c.v.}_{(ii)}$$

$$\underbrace{Body force}_{(iii)}+\underbrace{pressure force}_{(iv)}+\underbrace{viscous force}_{(v)} \quad (8.11)$$

$$\frac{\partial}{\partial t}(\rho \times \text{volume} \times \vec{v}) = \frac{\partial \rho \vec{v}}{\partial t}\delta x \, \delta y \times 1 = \left(\frac{\partial \rho u}{\partial t}, \frac{\partial \rho v}{\partial t}\right)\delta x \, \delta y \times 1$$

$$\dot{m}_3 \times \vec{v}_3 - \dot{m}_1 \times \vec{v}_1 + \dot{m}_4 \times \vec{v}_4 - \dot{m}_2 \times \vec{v}_2 \quad (8.12)$$

$$\vec{v}_1 = (u,v) - \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)\frac{\delta x}{2}, \vec{v}_2 = (u,v) + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)\frac{\delta x}{2},$$

$$\vec{v}_3 = (u,v) - \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)\frac{\delta y}{2}, \vec{v}_4 = (u,v) + \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)\frac{\delta y}{2}$$

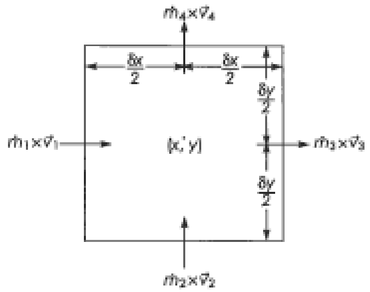

Fig. 8.14 Rectangular space of δx x δy

$$\rho\left(u + \frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\delta y \times 1\left(u + \frac{\partial u}{\partial x}\frac{\delta x}{2}\right) - \rho\left(u - \frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\delta y \times 1\left(u - \frac{\partial u}{\partial x}\frac{\delta x}{2}\right)$$

$$\rho\left(v + \frac{\partial v}{\partial y}\frac{\delta y}{2}\right)\delta x \times 1\left(u + \frac{\partial u}{\partial y}\frac{\delta y}{2}\right) - \rho\left(v - \frac{\partial v}{\partial y}\frac{\delta y}{2}\right)\delta x \times 1\left(u - \frac{\partial u}{\partial y}\frac{\delta y}{2}\right)$$

$$\rho\left(2u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right)$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+u\underbrace{\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right\}}_{0\ \text{Eqn}(2.46)}\right)\delta x\,\delta y\times 1 \qquad (8.13a)$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\delta x\,\delta y\times 1 \quad (8.13b)$$

$$\rho\delta x\delta y\times 1\times\vec{g}=\left(\rho g_x,\rho g_y\right)\delta x\delta y\times 1 \quad (8.14)$$

$$\left(p-\frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\times 1-\left(p+\frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\times 1=\frac{-\partial p}{\partial x}\delta x\,\delta y\times 1 \quad (8.15a)$$

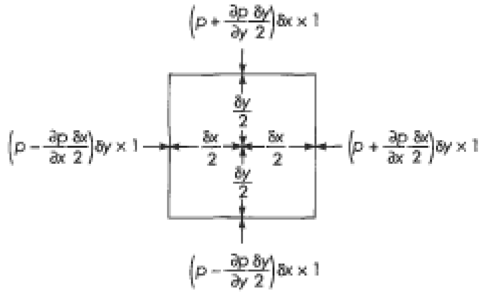

Fig. 8.15 Pressure forces acting on the infinitesimal control volume

$$\frac{-\partial p}{\partial y}\delta x\delta y\times 1 \quad (8.15b)$$

$$\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}\right)\delta x\,\delta y\times 1 \quad (8.16a)$$

$$\left(\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}\right)\delta x\,\delta y\times 1 \quad (8.16b)$$

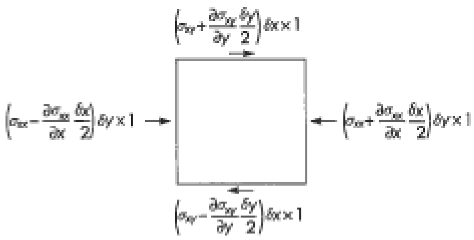

Fig. 8.16 x-component of forces due to viscous stress acting on infinitesimal control volume

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y} \quad (8.17a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y} \quad (8.17b)$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z} \quad (8.18a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z} \quad (8.18b)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z} \quad (8.18c)$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho g_x-\frac{\partial p}{\partial x} \quad (8.19a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho g_y-\frac{\partial p}{\partial y} \quad (8.19b)$$

Control questions

1. Describe the velocity and acceleration of two-dimensional flow in Cartesian and polar coordinates.
2. Describe the equation of continuity or conservation of mass for the case of two-dimensional flow.
3. Describe the equation of continuity in polar coordinates for the case of two-dimensional flow.
4. Describe the stream function and streamline for the case of two-dimensional flow.
5. Describe velocity components in terms of ψ in Cartesian and polar coordinates.
6. Describe the momentum equation for the case of two-dimensional flow.
7. Describe the Euler equations for the case of two-dimensional flow.

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Lecture 9. The Navier-Stokes equations

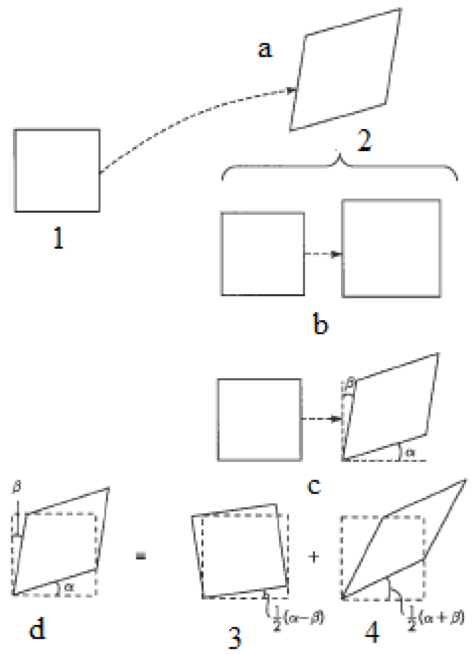

Fig. 9.1 Transformation of a fluid element as it moves through the flow field:
1 – start; 2 – end; 3 – rotation; 4 – shear;
a) translation; b) dilation; c) distortion; d) distortion

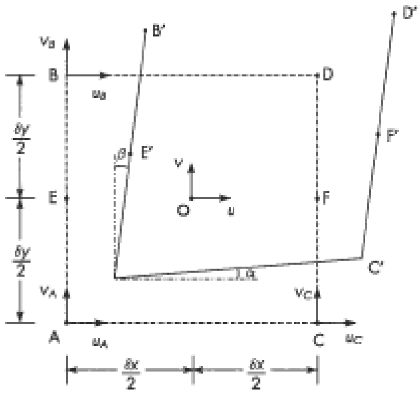

Fig. 9.2 Shear strain of elemental control volume ABCD

$$u_A = u - \frac{\partial u}{\partial x}\frac{\delta x}{2} - \frac{\partial u}{\partial y}\frac{\delta y}{2}, v_A = v - \frac{\partial v}{\partial x}\frac{\delta x}{2} - \frac{\partial v}{\partial y}\frac{\delta y}{2} \quad (9.1a)$$

$$u_B = u - \frac{\partial u}{\partial x}\frac{\delta x}{2} + \frac{\partial u}{\partial y}\frac{\delta y}{2}, v_B = v - \frac{\partial v}{\partial x}\frac{\delta x}{2} + \frac{\partial v}{\partial y}\frac{\delta y}{2} \quad (9.1b)$$

$$u_C = u + \frac{\partial u}{\partial x}\frac{\delta x}{2} - \frac{\partial u}{\partial y}\frac{\delta y}{2}, v_C = v + \frac{\partial v}{\partial x}\frac{\delta x}{2} - \frac{\partial v}{\partial y}\frac{\delta y}{2} \quad (9.1c)$$

$$x_{A'} = u_A \delta t, y_{A'} = v_A \delta t etc \quad (9.2)$$

$$\alpha=\frac{y_{C'}-y_{A'}}{\delta x}=\left(v_C-v_A\right)\frac{\delta t}{\delta x}=\left\{v+\frac{\partial v}{\partial x}\frac{\delta x}{2}-\frac{\partial v}{\partial y}\frac{\delta y}{2}-\left(v-\frac{\partial v}{\partial x}\frac{\delta x}{2}-\frac{\partial v}{\partial y}\frac{\delta y}{2}\right)\right\}\frac{\delta t}{\delta x}=\frac{\partial v}{\partial x}\delta t \quad (9.3\text{ a})$$

$$\beta=\frac{x_{B'}-x_{A'}}{\delta y}=\left(u_B-u_A\right)\frac{\delta t}{\delta y}=\left\{u+\frac{\partial u}{\partial x}\frac{\delta x}{2}-\frac{\partial u}{\partial y}\frac{\delta y}{2}-\left(u-\frac{\partial u}{\partial x}\frac{\delta x}{2}-\frac{\partial u}{\partial y}\frac{\delta y}{2}\right)\right\}\frac{\delta t}{\delta y}=\frac{\partial u}{\partial y}\delta t \quad (9.3\text{b})$$

$$\frac{d\gamma_{xy}}{dt}=\frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)=\left(\frac{\partial v}{\partial x}\delta t+\frac{\partial u}{\partial x}\delta t\right)\frac{1}{2\delta t}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \quad (9.4\text{a})$$

$$\frac{d\gamma_{xz}}{dt}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\frac{d\gamma_{yz}}{dt}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \quad (9.4\text{b,c })$$

$$\varepsilon_{xx}=\frac{x_{F'}-x_{E'}}{x_F-x_E}=\frac{\left(x_{F'}-x_{E'}\right)\delta t}{\delta x}=\left\{u+\frac{\partial u}{\partial x}\frac{\delta x}{2}-\left(u+\frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\right\}\frac{\delta t}{\delta x}=\frac{\partial u}{\partial x}\delta t \quad (9.5)$$

$$\frac{d\varepsilon_{xx}}{\delta t}=\frac{\partial u}{\partial x},\frac{d\varepsilon_{yy}}{\delta t}=\frac{\partial v}{\partial y},\frac{d\varepsilon_{zz}}{\delta t}=\frac{\partial w}{\partial z} \qquad (9.6\text{ a,b, c})$$

$$\begin{pmatrix}\dot{\varepsilon}_{xx} & \dot{\gamma}_{xy}\\ \dot{\gamma}_{yx} & \dot{\varepsilon}_{yy}\end{pmatrix} \qquad (9.7)$$

$$\zeta=\frac{d\alpha}{dt}-\frac{d\beta}{dt}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad (9.8)$$

$$\Omega=(\xi,\eta,\zeta)=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \quad (9.9\text{ a, b, c})$$

$$\Omega=\nabla\times v \quad (9.10)$$

$$\zeta=\frac{q_t}{r}+\frac{\partial q_t}{\partial r}-\frac{1}{r}\frac{\partial q_n}{\partial\theta} \quad (9.11)$$

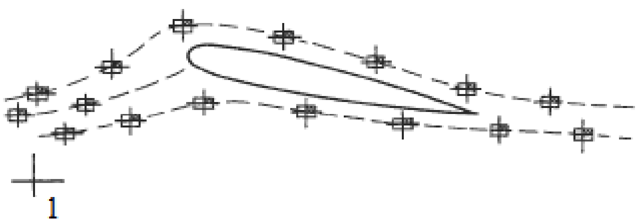

Fig. 9.3 state of pure translation
1 - Reference axes

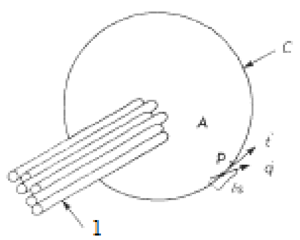

Fig. 9.4
1 - Bundle of vortex tubes

$$\Gamma = \iint_A n \cdot \Omega \, dA \quad (9.12)$$

$$\Gamma = \iint_A \zeta dA \quad (9.13)$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = 0 \quad (9.14)$$

$$\Gamma = \iint_A n \cdot \Omega \, dA = \iint_A n \cdot \nabla \times q dA = \oint_c q \cdot t ds \quad (9.15)$$

Stress $\propto$ Strain (9.16)

Stress $\propto$ Rate of strain (9.17)

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = 2\mu \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\gamma}_{xy} \\ \dot{\gamma}_{yx} & \dot{\varepsilon}_{yy} \end{pmatrix} \quad (9.18)$$

$$\lambda \begin{pmatrix} \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} & 0 \\ 0 & \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} \end{pmatrix} \quad (9.19)$$

$3\lambda + 2\mu = 0$ or $\lambda = 2/3\ \mu$

$\mu' = \lambda + 2/3\mu \approx 0$ (9.20)

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}, \sigma_{yy} = 2\mu\frac{\partial v}{\partial y}, \sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (9.21)$$

$$g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$¿ g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mu\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{¿0, \text{Eqn}(9.24)} \quad (9.22)$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (9.23a)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (9.23b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (9.24)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (9.25)$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (9.26a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \quad (9.26b)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad (9.26c)$$

$$\rho\frac{Du}{Dt} \text{ where } \frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \quad (9.27)$$

X=x/L, Y=y/L, Z=z/L, and T=tU/L (9.28)

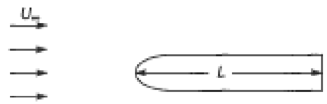

Fig. 9.6 Air flowing at speed U_∞ towards a body

$U = u/U_\infty$, $V = v/U_\infty$, $W = w/U_\infty$, $P = p/(\rho U^2_\infty)$ (9.29)

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0 \quad (9.30)$$

$$\frac{DU}{DT}=\frac{-\partial P}{\partial X}+\frac{1}{R}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right) \quad (9.31a)$$

$$\frac{DV}{DT}=\frac{-\partial P}{\partial Y}+\frac{1}{R}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial^2 V}{\partial Z^2}\right) \quad (9.31b)$$

$$\frac{DW}{DT}=\frac{-\partial P}{\partial Z}+\frac{1}{R}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}+\frac{\partial^2 W}{\partial Z^2}\right) \quad (9.31c)$$

$Re = \rho U_\infty L/\mu$ (9.32)

$\mu\, \partial^2u/\partial y^2 = 0$ implying $u = C_1y + C_2$ (9.33)

$u = U_T\, y/h = T/\mu\, y$ (9.34)

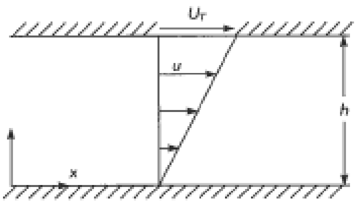

Fig. 9.7 Couette flow

Control questions.
1. Describe the deformation of the liquid element in the flow field.
2. Describe the shear strain rate and the strain rate.
3. Describe vortex formation and vortex formation in polar coordinates.
4. Describe the rotational and non-rotational flows, circulation.
5. Derive the Navier-Stokes equation.
6. Describe the properties of the Navier-Stokes equations.
7. Describe the exact solutions of the Navier-Stokes equations - the Couette flow.

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Lecture 10. Potential flow

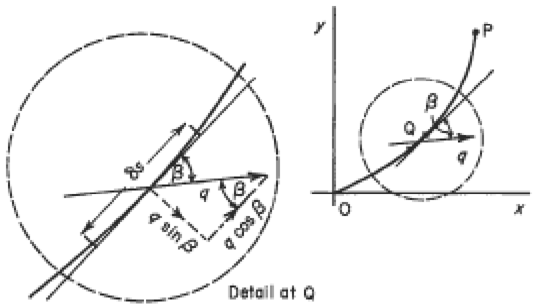

Fig. 10.1 General two-dimensional fluid flow

$$\phi = \int_{OP} q\cos\beta\delta s \quad (10.1)$$

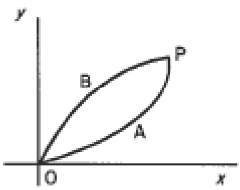

Fig. 10.2 Tangential flow component from O to P via A

$$\phi = \int_{OP} q\cos\beta ds = \int_{OP_1} q\cos\beta ds = \int_{OPP_1} q\cos\beta ds,$$

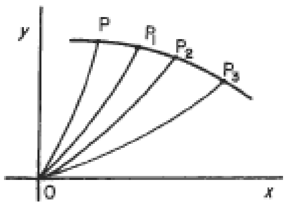

Fig. 10.3 Flow along OP

$$\delta\phi = u\delta x + v\delta y$$

$$\delta\phi = \partial\phi/\partial x\ \delta x + \partial\phi/\partial y\ \delta y$$

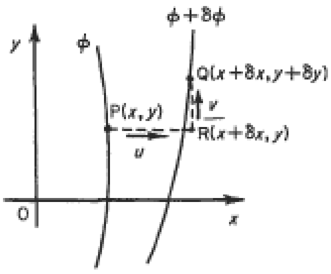

Fig. 10.4 Points on equipotentials in Cartesian coordinates

$$\begin{pmatrix} u=\dfrac{\partial\phi}{\partial x} \\ v=\dfrac{\partial\phi}{\partial y} \end{pmatrix} \quad (10.2)$$

$\delta\phi = q_n\delta r + q_t(r + \delta r)\delta\theta = q_n\delta r + q_t r\delta\theta$

$$\delta\phi = \partial\phi/\partial r\ \delta r + \partial\phi/\partial\theta\ \delta\theta$$

$$\begin{pmatrix} q_n=\dfrac{\partial\phi}{\partial r} \\ q_t=\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} \end{pmatrix} \quad (10.3)$$

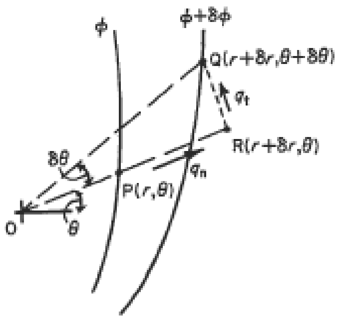

Fig. 10.5 Points on equipotentials in polar coordinates

$q = \partial\phi/\partial s$

$\partial u/\partial x + \partial v/\partial y = 0$ (i)

$\partial v/\partial x + \partial u/\partial y = \zeta$ (ii)

$u = \partial\psi/\partial y \quad v = \partial\psi/\partial x$ (iii)

$u = \partial\psi/\partial x \quad v = \partial\psi/\partial y$ (iv)

$$\partial^2\psi/\partial x\partial y - \partial^2\psi/\partial x\partial y = 0$$

$$\partial^2\phi/\partial x\partial y - \partial^2\phi/\partial x\partial y = 0$$

$$\partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2 = 0 = \partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 \quad (10.4)$$

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

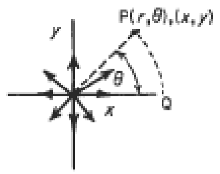

Fig. 10.6 Stream function ψ of a source

ψ = flow across OQ + flow across QP

= velocity across OQ x OQ + velocity across QP x QP

$= 0 + m/2\pi r \times r\theta$

$\psi = m\theta/2\pi$

$\theta = m/2\pi \tan^{-1} y/x$ (10.5)

$0 \leq \theta \leq 2\pi$.

$\psi = -(m/2\pi)\theta$

$$\phi = \int_{OQ} q\cos\beta ds + \int_{OP} q\cos\beta ds$$

$$\int_{OQ} q_n dr + \int_{OP} q_t r d\theta = \int_{OQ} q_n dr + 0$$

$q_n = m/2\pi r$

$$\phi = \int_{r_0}^{r} \frac{m}{2\pi r} dr = \frac{m}{2\pi} \ln \frac{r}{r_0} \quad (10.6)$$

$$q_n = m/2\pi r = \partial\phi/\partial r$$

$$\phi = \int_{r_0}^{r} \frac{m}{2\pi r} dr = \frac{m}{2\pi} \ln \frac{r}{r_0} \quad (10.7)$$

$\phi = m/4\pi \ln(x^2 + y^2)$ (10.8)

$\phi = m/2\pi \ln r - C$

$C = m/2\pi \ln r_0$

$r = e^{2\pi(\phi + C)/m}$ (10.9)

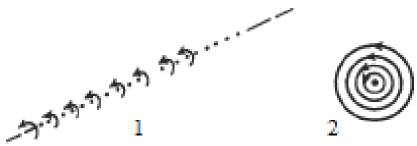

Fig. 10.7 Straight line vortex
1 - Straight line vortex; 2 - Cross-section showing a few of the associated streamlines

$\zeta = q_t/r + dq_t/dr = 0$, i.e. $1/r\, d/dr\, (rq_t) = 0$

$rq_t = C$

$$\Gamma = \oint \vec{q} \cdot \vec{t}\, ds$$

$\Gamma = 2\pi r q_t = 2\pi C$.

$q_t = d\psi/dr = \Gamma/2\pi r$

$$\psi = \int \frac{-\Gamma}{2\pi r} dr$$

$$\psi = -\int_{r_0}^{r} \frac{\Gamma}{2\pi r} dr$$

$$\psi = -\left[\frac{\Gamma}{2\pi} \ln r\right]_{r_0}^{r} = \frac{-\Gamma}{2\pi} \ln \frac{r}{r_0} \quad (10.10)$$

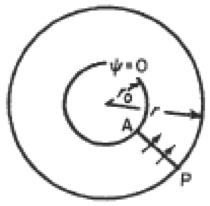

Fig. 10.8 Vortex located at the origin of a polar system of coordinates

$q_n = 0$, ϕ is a function of θ, and

$q_t = 1/r\ d\phi/d\theta = \Gamma/2\pi r$

$d\phi = \Gamma/2\pi\ d\theta$

$\phi = \Gamma/2\pi\ \theta + \text{constant}$

$\phi = \Gamma/2\pi\ \theta$ (10.11)

$\psi = m\theta/2\pi$ (Eqn(10.5))

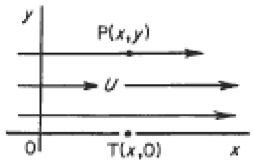

Fig. 10.9 Flow streaming past the coordinate axes Ox, Oy at velocity U parallel to Ox

ψ = flow across line OTP

= flow across line OT plus flow across line TP

= 0 + U x length TP

$= 0 + U_y$

$\psi = U_y$ (10.12)

$\psi = \text{constant} = U_y$

$y = \psi/U$ = constant on streamlines

ϕ = flow along contour OTP

= flow along OT + flow along TP

$= U_x + o$

$\phi = U_x$ (10.13)

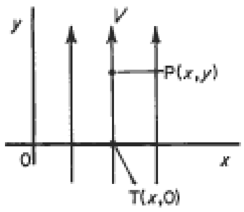

Fig. 10.10 Flow of constant velocity parallel to 0y axis

ψ = flow across OT + flow across TP

$= -V_x + 0$

$\psi = - V_x$ (10.14)

ϕ = flow along OT + flow along TP

$= 0+V_y$

$\phi = V_y$ (10.15)

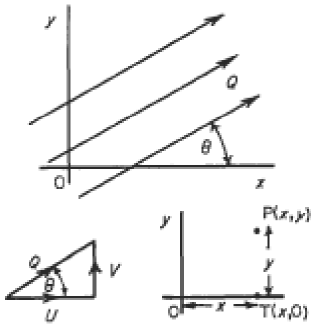

Fig. 10.11 Flow of constant velocity in any direction

ψ = flow across OT (going right to left, therefore negative in sign)

+flow across TP

=-component of Q parallel to Oy times x

+component of Q parallel to Ox times y

$\psi =-V_x+ U_y$ (10.16)

$-Vx + Uy$ = constant

ϕ = flow along OT + flow along TP

$= Ux + Vy$

$\phi = Ux + Vy$ (10.17)

Control questions

1. Describe the potential flow and velocity potential.
2. Describe the equipotential.
3. Describe the components of velocity on the basis of ϕ.
4. Describe the Laplace equation.
5. Describe standard flows based on ψ and ϕ.
6. Describe a linear (point) vortex and uniform flow.
7. Describe the flows of constant velocity parallel to the axis 0y, and constant velocity in any direction.

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Lecture 11. Standard flows in terms of ψ and ϕ

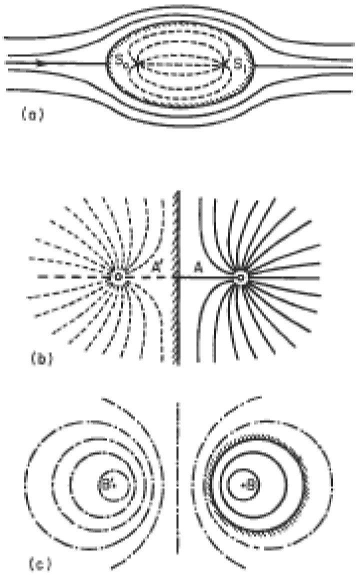

Fig. 11.1 Image systems

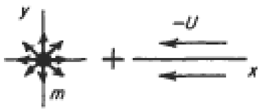

Fig. 11.2 A source in a uniform horizontal stream

$\psi = m\theta/2\pi - U_y$ (11.1)

$\psi = m/2\pi \tan^{-1} y/x - U_y$ (11.2)

$\phi = m/2\pi \ln r/r_0 - U_x$

$\phi = m/2\pi \ln(x^2/r^2_0 + y^2/r^2_0) - U_x$

$\phi = m/2\pi \ln r/r_0 - U_r \cos\theta$

$\psi = m\theta_1/2\pi - m\theta_2/2\pi = m/2\pi\,(\theta_1 - \theta_2)$

$\psi = m/2\pi\,\beta$ (11.3)

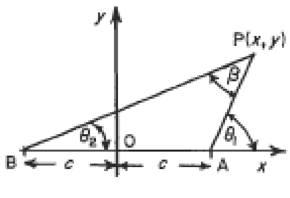

Fig. 11.3 Source-sink pair

$\tan\theta_1 = y/(x - c)$, $\tan\theta_2 = y/(x + c)$

$\tan(\theta_1 - \theta_2) = (\tan\theta_1 - \tan\theta_2)/(1 + \tan\theta_1 \tan\theta_2) = [y/(x - c) - y/(x + c)]/[1 + y^2/(x^2 - c^2)]$

$\beta = \theta_1 - \theta_2 = \tan^{-1} 2cy/(x^2 + y^2 - c^2)$

$\psi = m/2\pi \tan^{-1} 2cy/(x^2 + y^2 - c^2)$ (11.4)

$\tan(2\pi/m\ \psi) = 2cy/(x^2 + y^2 - c^2)$

$x^2 + y^2 - c^2 = 2cy/\tan(2\pi\psi/m)$

$x^2 + y^2 - 2c \cot(2\pi\psi/m\ y) - c^2 = 0$

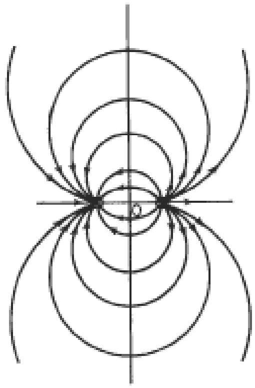

Fig. 11.4 Streamlines due to a source and sink pair

$$\phi = m/2\pi \ln r_1/r_0 - m/2\pi \ln r_2/r_0 = m/2\pi \ln r_1/r_2$$

$r^2_1 = (x- c)^2 + y^2 = x^2 + y^2 + c^2 - 2xc$

$r^2_2 = (x + c)^2 + y^2 = x^2 + y^2 + c^2 + 2xc$

$\phi = m/4\pi \ln(x^2 + y^2 + c^2 - 2xc)/(x^2 + y^2 + c^2 + 2xc)$

$e^{4\pi\phi/m} = (x^2 + y^2 + c^2 - 2xc)/(x^2 + y^2 + c^2 + 2xc) = \lambda$

$(x^2 + y^2 + c^2 + 2xc)\lambda = x^2 + y^2 + c^2 - 2xc$

$(x^2 + y^2 + c^2)(\lambda - 1) + 2xc(\lambda + 1) = 0$

$x^2 + y^2 + 2xc[(\lambda + 1)/(\lambda - 1)] + c^2 = 0$

$x = -c[(\lambda + 1)/(\lambda - 1)], \quad y = 0$

$x = -c[e^{4\pi\phi/m} +1]/[e^{4\pi\phi/m} - 1] = -c \coth 2\pi\phi/m, y = 0$

$$c\sqrt{\left(\frac{\lambda+1}{\lambda-1}\right)^2 - 1} = 2c\frac{\sqrt{\lambda}}{\lambda-1} = 2c\frac{e^{2\pi\phi/m}}{e^{4\pi\phi/m}-1}$$

$= 2c \operatorname{cosech} 2\pi\phi/m$

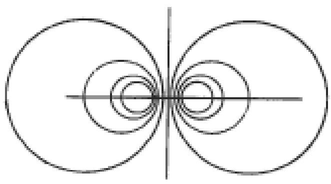

Fig. 11.5 Equipotential lines due to a source and sink pair

$$\psi = \mu/2\pi r \sin\theta - U_y \quad (11.5)$$

$\psi = \mu/2\pi\ y/(x^2 - y^2) - U_y$ (11.6)

$$y(\mu/2\pi(x^2 - y^2) - U) = 0$$

$y=0$ or $x^2 - y^2 = \mu/2\pi U$

$\psi = \mu/2\pi r \sin\theta - Ur \sin\theta$

$$\psi = \sin\theta\ (\mu/2\pi r - Ur) = 0 \text{ for } \psi = 0$$

$$\sin\theta = 0 \text{ so } \theta = 0 \text{ or } \pm\pi$$

$$\mu/2\pi r - Ur = 0 \text{ giving } r = \sqrt{\frac{\mu}{2\pi U}} = a$$

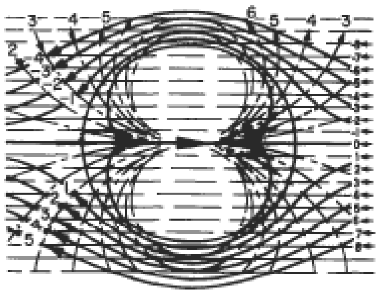

Fig. 11.6 Streamlines due to a doublet in a uniform stream

$\phi = -\,Ur\cos\theta + \mu/2\pi r\cos\theta$

$\phi = -\,U\cos\theta\,(r + a^2/r)$

$\psi = \mu/2\pi r\sin\theta - Ur\sin\theta$

$\psi = U\sin\theta\,(\mu/2\pi rU - r)$

$\psi = U\sin\theta\,(a^2/r - r)$ (11.6)

$$\left(\begin{array}{c} q_n = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = U\cos\theta\left(\dfrac{a^2}{r^2} - 1\right) \\ q_t = \dfrac{\partial\psi}{\partial r} = U\sin\theta\left(\dfrac{a^2}{r^2} + 1\right) \end{array}\right) \quad (11.7)$$

$$\text{(i) } q_n = U\cos\theta\,[1 - 1] = 0$$

$$\text{(ii) } q_t = U\sin\theta\,[1 + 1] = 2U\sin\theta.$$

$$q = 2U\sin\theta$$

$$p_0 + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho q^2 = p + \tfrac{1}{2}\rho\,(2U\sin\theta)^2$$

$$p - p_0 = \tfrac{1}{2}\rho U^2\,[1 - 4\sin^2\theta]$$

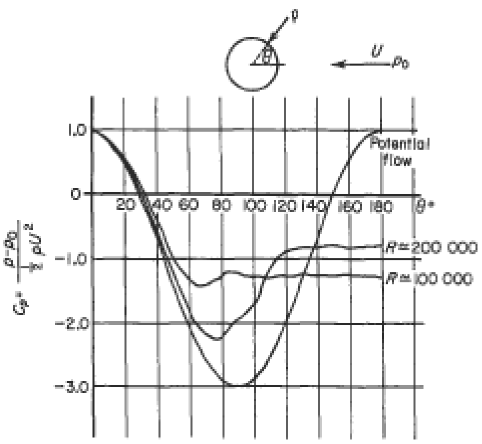

Fig. 11.7

$$\psi = \mu/2\pi r\sin\theta - U_y - \Gamma/2\pi\ln(r/r_0)$$

$\psi = U_r \sin\theta(\mu/2\pi r^2 U - 1) - \Gamma/2\pi \ln(r/r_0)$

$$\psi = U_r \sin\theta(a^2/r^2 - 1) - \Gamma/2\pi \ln(r/a)$$

$$\begin{pmatrix} q_t = \dfrac{-\partial\psi}{\partial r} = U\sin\theta\left(\dfrac{a^2}{r^2}+1\right)+\dfrac{\Gamma}{2\pi r} \\ q_n = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = U\cos\theta\left(\dfrac{a^2}{r^2}-1\right) \end{pmatrix}$$

$$q=\sqrt{q_n^2+q_t^2}$$

$q_n = 0$

$q_t = 2U \sin\theta + \Gamma/2\pi a$ (11.48)

$$q = q_t = 2U \sin\theta + \Gamma/2\pi a$$

$$p_0 + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho q^2 = p + \tfrac{1}{2}\rho (2U \sin\theta + \Gamma/2\pi a)^2$$

$$p - p_0 = \tfrac{1}{2}\rho U^2 [1 - (2 \sin\theta + \Gamma/2\pi U a)^2] \quad (11.8)$$

$p_T - p_0 = \tfrac{1}{2}\rho U^2 (1 - [2 + B]^2) = -\tfrac{1}{2}\rho U^2 (3 + 4B + B^2)$ (11.9)

$p_B - p_0 = -\tfrac{1}{2}\rho U^2 (3 + 4B + B^2)$ (11.10)

$p_B - p_0 = 8B (-\tfrac{1}{2}\rho U^2) = -2/\pi a \rho UT$

$$l=\int_0^{2\pi}\frac{-1}{2}\rho U^2 a\left[1-\left(2\sin\theta+B^2\right)\right]\sin\theta d\theta$$

$$l=\frac{-1}{2}\rho U^2 a\int_0^{2\pi}\left[\sin\theta\left(1-B^2\right)-4B\sin^2\theta-4\sin^3\theta\right]d\theta$$

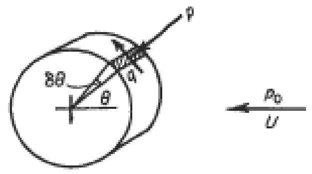

Fig. 11.8 The pressure and velocity on the surface of unit length of a cylinder of radius a

$$\int_0^{2\pi} 4B\sin^2\theta\, d\theta = 4B\pi$$

$l = \tfrac{1}{2}\rho U^2 a^4 B\pi$

$l = \rho UT$

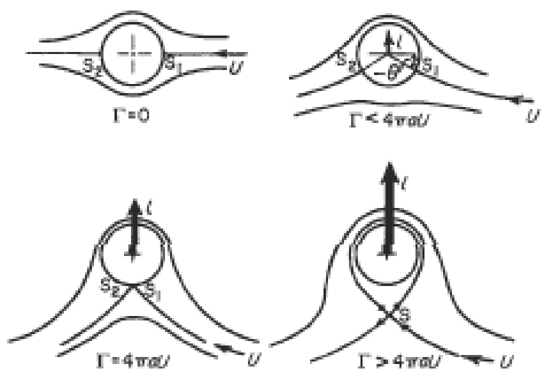

Fig. 11.9 The flow pattern around a spinning cylinder

$q_t = 2U \sin \theta + \Gamma/2\pi = 0$

$\theta = \arcsin(-\Gamma/2\pi aU)$

$(p + \partial p/\partial r\ \partial r/2)\ (r + \delta r/2)\ \delta\theta - (p - \partial p/\partial r\ \partial r/2)\ (r + \delta r/2)\ \delta\theta - p\ \delta r\ \delta\theta$

$\partial p/\partial r\ r\ \delta r\ \delta\theta$ (11.11)

$= pr\ \delta r\ \delta\theta\ q^2_t/r$ (11.12)

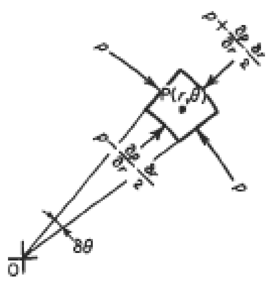

Fig. 11.10 Segmental element

$\partial p/\partial r = \rho q^2_t/r$ (11.13)

$\partial H/\partial r = \partial(p + \frac{1}{2}\rho q^2_t)/r = \partial p/\partial r + \rho q_t\ \partial q_t/\partial r$

$\partial H/\partial r = \rho\ q^2_t/r + pq_t\ \partial q_t/\partial r = pq_t\ (q_t/r + \partial q_t/\partial r)$

$\partial H/\partial r = \rho q_t\ \zeta$

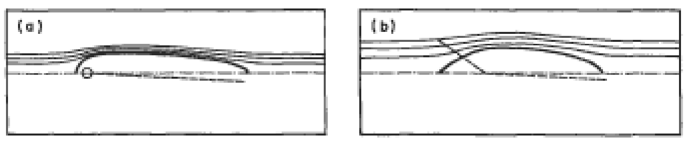

Fig. 11.11 Two examples of flow around bodies of revolution generated by (a) a point source plus a linear distribution of source strength; and (b) two linear distributions of source strength. The source distributions are denoted by broken lines

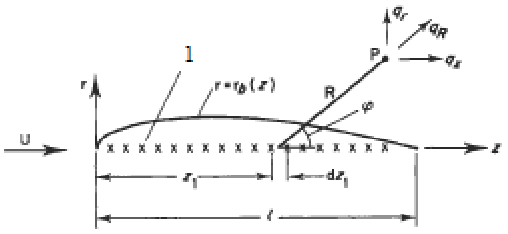

Fig. 11.12 Flow over a slender body of revolution modelled by source distribution
1 - Sources

$$q_R = \sigma(z_1)/4\pi R^2 \, dz_1 \quad (11.14)$$

$$q_r = \int_0^l q_R \sin\varphi \, dz_1 = \frac{1}{4\pi}\int_0^l \sigma(z_1)\frac{r}{\left[(z-z_1)^2+r^2\right]^{3/2}} dz_1$$

$$q_z = \int_0^l q_R \cos\varphi \, dz_1 = \frac{1}{4\pi}\int_0^l \sigma(z_1)\frac{z-z_1}{\left[(z-z_1)^2+r^2\right]^{3/2}} dz_1$$

$2\pi r \, q_r \, dz_1 = \sigma(z_1) \, dz_1$ at $r = r_b$ provided $r_b \to 0$

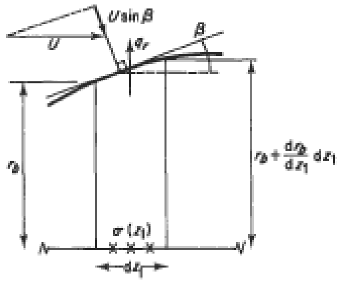

Fig. 11.13 Element of the body

$$\underbrace{q_r}_{\text{Sources}} = \underbrace{U\sin\beta = U\frac{dr_b}{dz_1}}_{\text{Oncomingflow}}$$

$$\sigma(z) = U \, dS/dz \quad (11.15)$$

$$q_z = \frac{1}{4\pi}\int_0^l \frac{\sigma(z_1)}{(z-z_1)^2} dz_1 \quad (11.16)$$

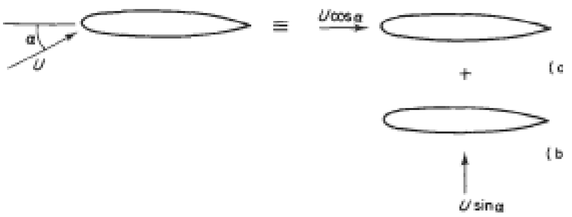

Fig. 11.14 Flow at angle of yaw around a body of revolution as the superposition of two flows

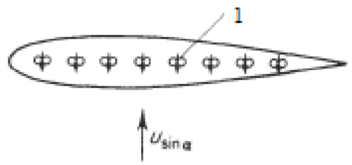

Fig. 11.15 Cross-flow over slender body of revolution modelled as distribution of doublets:
1 - Doublet

Control questions

1. Describe the systems of imaging the contours of currents.
2. Describe the source in a homogeneous horizontal flow.
3. Give a description of the source-stock money.
4. Describe the equipotential lines of the source and drain pairs
5. Describe the flow around a circular cylinder defined by a dipole in a uniform horizontal flow.
6. Describe the pressure distribution around the cylinder.
7. Give a description of a rotating cylinder in a homogeneous flow.
8. Give a description of the normal force on a rotating circular cylinder in a homogeneous flow.
9. Give a description of the flow scheme around the rotating cylinder.
10. Give a description of the Bernoulli equation for the rotational flow.
11. Describe the flow around thin bodies.

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Lecture 12. Generation of lift

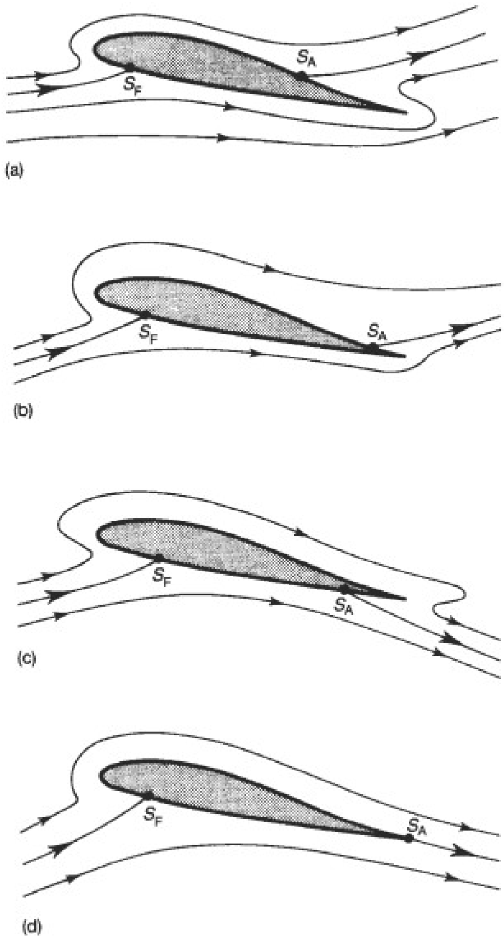

Fig. 12.1 Effect of circulation on the flow around an aerofoil at an angle of incidence
(a) No circulation; (b) Low circulation; (c) High circulation; (d) Circulation such that Kutta condition is satisfied

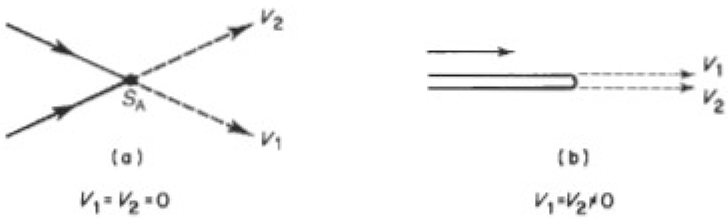

Fig. 12.2 Practical aerofoils (a) and infinitely thin aerofoils (b)

$$\phi_A - \phi_B = \int_{AB} q\cos\alpha ds \quad (12.1)$$

$$\phi_A - \phi_B = \int_{AB} (udx + vdy)$$

$\Gamma = \oint q\cos\alpha ds \vee \Gamma = \oint (udx+vdy)$

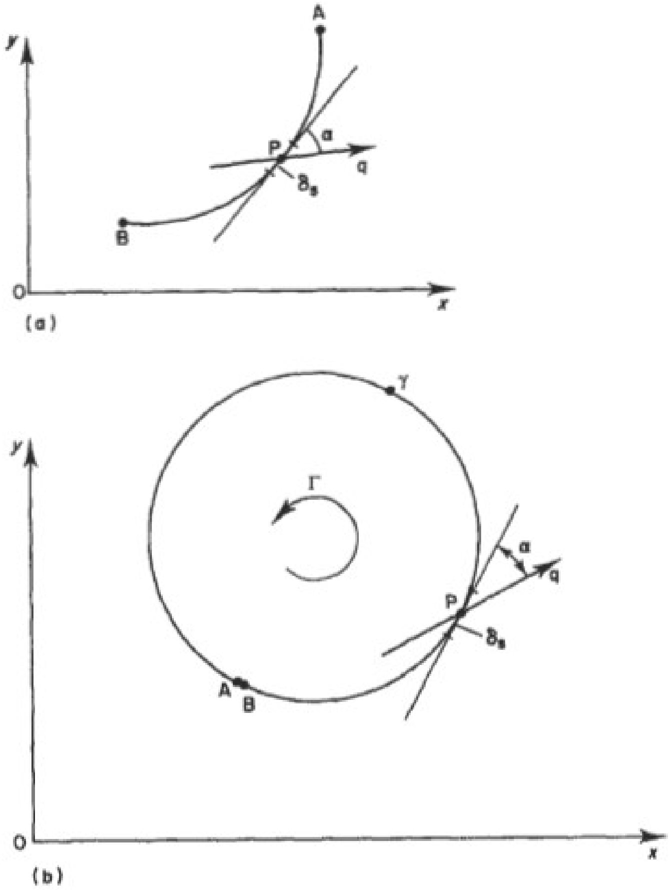

Fig. 12.3 (a) An open curve in a potential flow. (b) A closed curve in a circulatory flow; A and B coincide

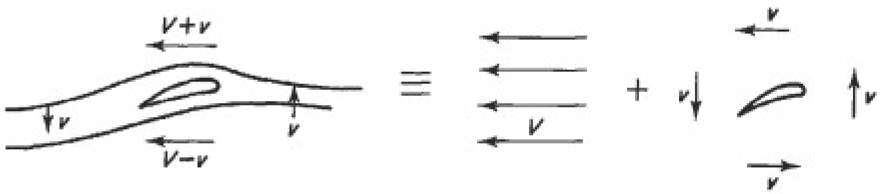

Fig. 12.4 Uniform irrotational portion and a circulating portion

$\Gamma = \phi_{CA} \pm \phi_{CB} = 0$

$\Delta\Gamma = (v + \partial y/\partial x\ \delta x/2)\delta y - (u + \partial u/\partial y\ \delta y/2)\delta x - (v - \partial v/\partial x\ \delta x/2)\delta y + (u - \partial u/\partial y\ \delta y/2)\delta x$
$\Delta\Gamma = (\partial v/\partial x - \partial u/\partial y)\delta x\delta y$

$$\Gamma = \underbrace{\iint\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}_{\text{over the area}} = \underbrace{\oint(u\,\mathrm{d}x + v\,\mathrm{d}y)}_{\text{round the circuit}}$$

$\partial v/\partial x - \partial u/\partial y = \zeta$

$\Gamma = \zeta \text{ x } \delta x \delta y = \zeta$ x area of element

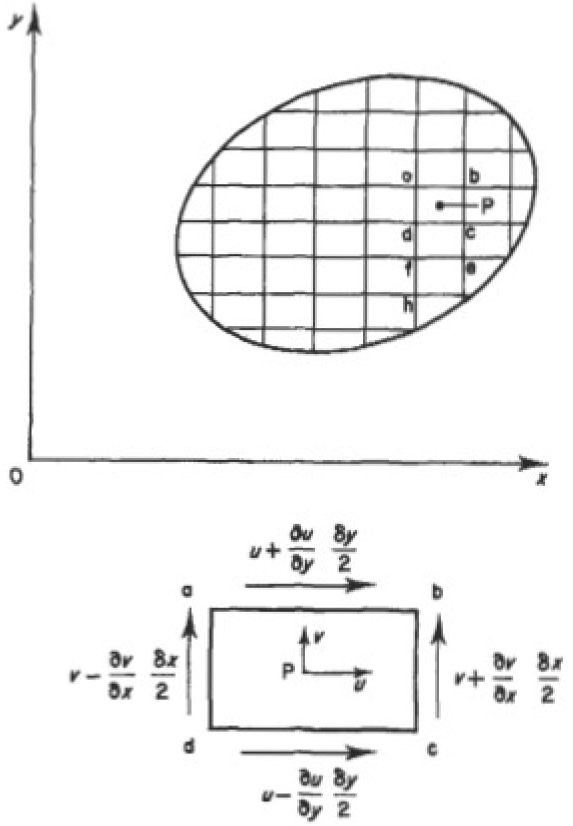

Fig. 12.5 Circuit of integration

$$vorticity = \lim_{area \to 0} \frac{\Gamma}{area\, of\ circuit} \quad (12.3)$$

$q = C / r_1$

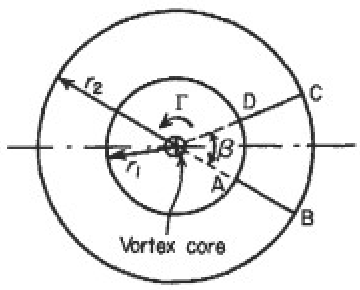

Fig. 12.6 Two circuits in the flow around a point vortex
1 – vortex core

$$\Gamma = \oint \frac{C}{r_1} ds$$

$ds = r_1 d\theta$

$$\Gamma = \int_0^{2\pi} \frac{C}{r_1} r_1 \, d\theta = 2\pi C \quad (12.4)$$

$\Gamma = 2\pi q r$

$q = \Gamma / 2\pi r$ (12.5)

$$\delta\Gamma = \int_{HC}^{\square} q \cos\alpha \, ds = \int_0^{\beta} q r_2 \, d\theta$$

$q = \Gamma / 2\pi r_2$

$$\delta\Gamma = \int_0^{\beta} \frac{\Gamma}{2\pi r_1} (-1) r_1 \, d\theta = \frac{-\beta\Gamma}{2\pi}$$

$\Gamma = 0 + \beta\Gamma/2\pi + 0 - \beta\Gamma/2\pi = 0$ (12.6)

$l = \rho V \Gamma$

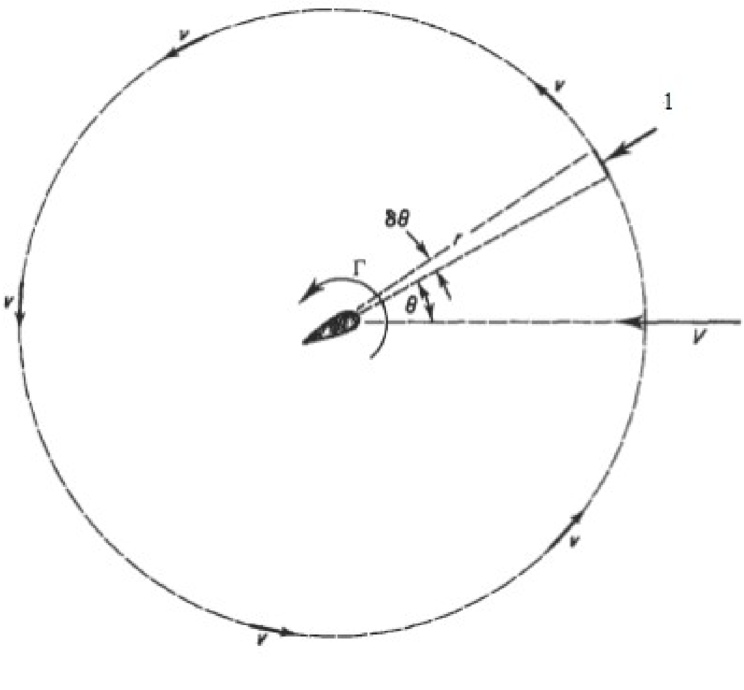

Fig. 12.7 Lift on any aerofoil
1 – static pressure p

$p_0 = \frac{1}{2} \rho V^2 = p + \frac{1}{2} \rho (V^2 + v^2 + 2Vv\sin\theta)$

$p = p_0 - \rho V v \sin\theta$

$-pr \sin\theta \, \delta\theta$

$$l_b = -\int_0^{2\pi} (p_0 - \rho V v \sin\theta) r \sin\theta \, d\theta = +\rho V v r \pi \quad (12.7)$$

$$l_i = +\int_0^{2\pi} \rho V v r \cos^2\theta d\theta = \rho V v r \pi \quad (12.8)$$

$l = 2\rho V v r \pi$ (12.9)

$v = \Gamma / 2\pi r$

$1 = \rho V \Gamma$ (12.10)

$Z = \zeta + C^2/\zeta$

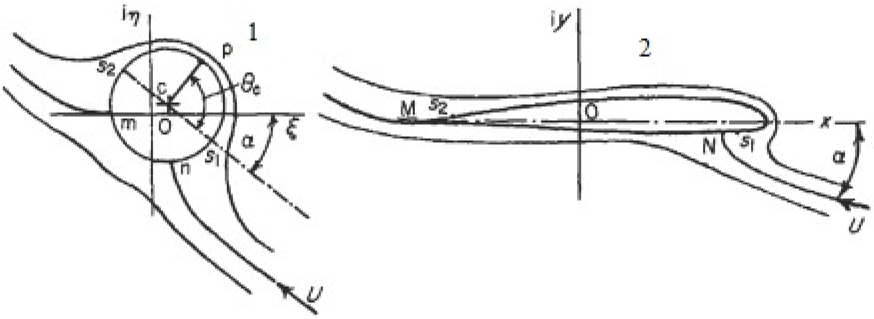

Fig. 12.8 Zhukovsky transformation, of the flow around a circular cylinder with circulation, to that around an aerofoil generating lift
1 - ζ plane; 2 - z plane

Fig. 12.9 Vortex distribution on the camber line

$u = U\cos\alpha + u'$. $v = U\sin\alpha + v'$

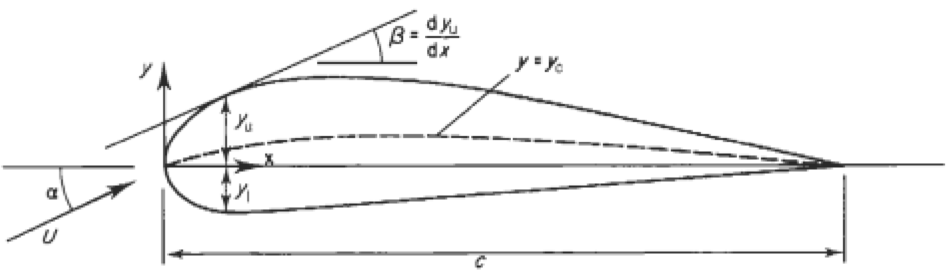

Fig. 12.10 Typical cambered aerofoil

$-u\sin\beta + v\cos\beta = 0$ at $y = y_u$ and y_1

$-(U\cos\alpha + u')dy/dx + U\sin\alpha + v' = 0$ at $y = y_u$ and y_1 (12.11)

y_u and $y_1 << c$; α, dy_u/dx and $dy_1/dx << 1$ y_u i $y_1 << c$; α, dy_u/dx i $dy_1/dx << 1$

u' and $v' << U$,

$v' = U\, dy_u/dx - U\alpha$ and $v' = U\, dy_1/dx - U\alpha$ (12.12)

$y_u=y_c+y_t$ and $y_u=y_c-y_t$

$$v' = \underbrace{U\frac{d\,y_c}{dx} - U\alpha}_{circulatory} \pm \underbrace{U\frac{d\,y_t}{dx}}_{non\,circulatory}$$

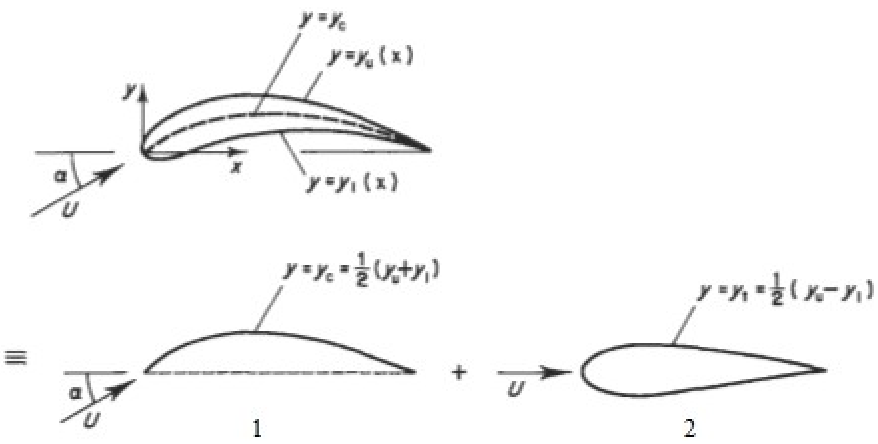

Fig. 12.11 Cambered thin aerofoil at incidence as superposition of a circulatory and non-circulatory flow
1 - Cumbered plate at incidence (circulatory flow); 2 - Symmetric aerofoil at zero incidence (non-circulatory flow)

$\Gamma=\int_0^c$ kds (12.13)

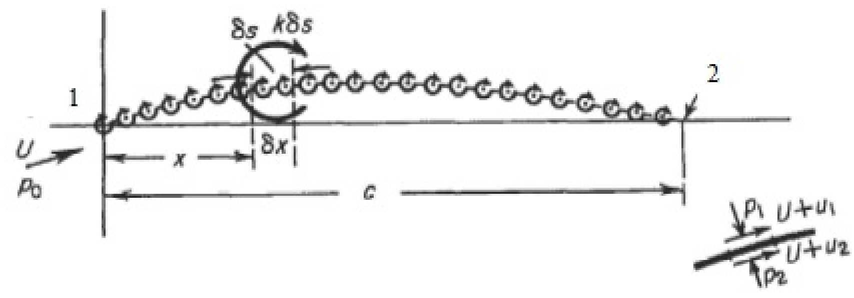

Fig. 12.12 Insert shows velocity and pressure above and below δs
1 – leading edge; 2 – trailing edge

$v' = U\, dy_c/dx - U\alpha$ at $y=0$, $0 \le x \le l$ (12.14)

$\Gamma=\int_0^c$ kdx (12.15)

$l=\rho U\Gamma=\rho U\int_0^c$ kdx (12.16)

$$l = \int_0^c \rho U k dx = \int_0^c p dx \quad (12.17)$$

$$M = -\int_0^c p x dx = -\rho U \int_0^c k x dx \quad (12.18)$$

$p_1 + ½ \rho (U+u_1)^2 = p_0 + ½ \rho U^2$
$p_2 + ½ \rho (U+u_2)^2 = p_0 + ½ \rho U^2$

$p_2 - p_1 = ½ \rho U^2 2[2 (u_1/U - u_2/U) + (u_1/U)^2 - (u_2/U)^2]$

$p = p_2 - p_1 = \rho U (u_1 - u_2)$ (12.19)

$k\delta x = +(U + u_1)\delta x - (U + u_2)\delta x = (u_1 - u_2)\delta x$ (12.20)

$k = 0$ at $x=c$ (12.21)

$1/2\pi \; k\delta x/(x - x_1)$

$$v' = \frac{1}{2\pi}\int_0^c \frac{k dx}{x - x_1}$$

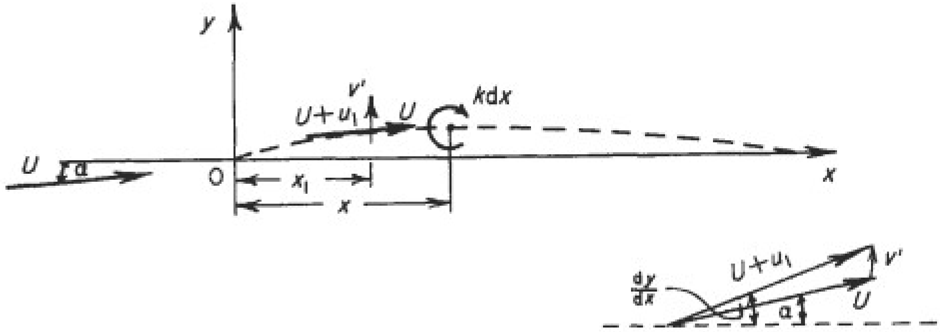

Fig. 12.13 Velocities at x_1 from 0: $U + u_1$, resultant tangential to camber lines; v', induced by chordwise variation in circulation; U, free stream velocity inclined at angle α to Ox

$$U\left[\frac{dy_c}{dx} - \alpha\right] = \frac{1}{2\pi}\int_0^c \frac{k dx}{x - x_1} \quad (12.22)$$

Control questions

1. Give a description of Kutt's condition.
2. Describe the circulation and vortices.
3. Describe the circulation and lifting force (Kutt-Zhukovsky theorem).
4. Give a description of the development of profile theory.
5. Give a description of the general theory of fine profile

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Lecture 13. Aerofoils

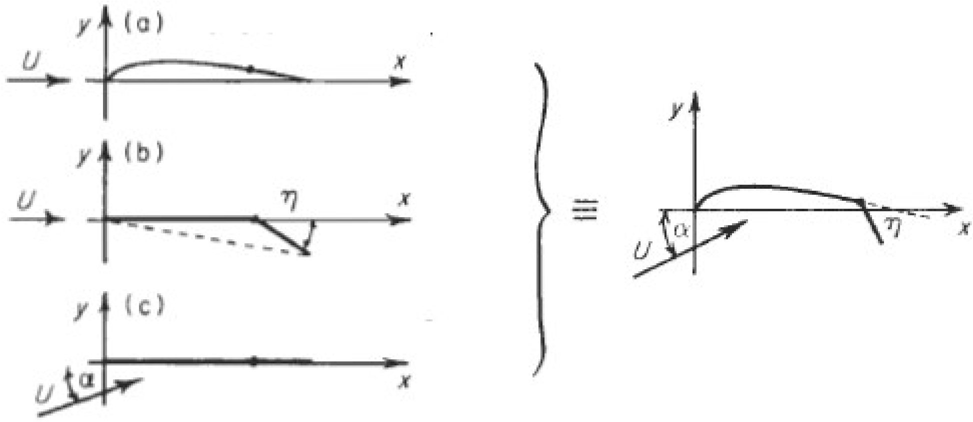

Fig. 13.1 Subdivision of lift contributions to total lift of cambered flapped aerofoil:
(a) Due to camber line shape; (b) Due to flap deflection; (c) Due to incidence change

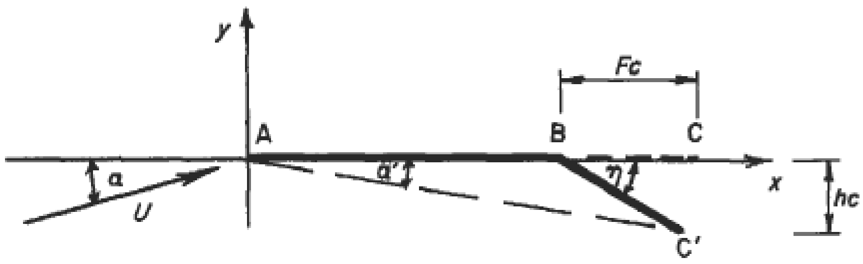

Fig. 13.2 Camber line made up of chord of aerofoil and flap chord

$$A_0 = \frac{1}{\pi}\int_0^{\pi} \frac{dy_c}{dx} d\theta$$

$$A_0 = \left\{ \frac{1}{\pi}\int_0^{\pi} 0 \, d\theta + \frac{1}{\pi}\int_0^{\pi} \frac{h}{F} d\theta \right\}$$

$A_0 = -(1 - \phi/\pi)\, h/F$

$A_0 = -(1 - \phi/\pi)\eta$

$$A_0 = \frac{2}{\pi}\int_0^{\phi} \frac{dy_c}{dx} \cos n\theta \, d\theta$$

$$A_n = \frac{2}{\pi}\left\{ \int_0^{\phi} 0 \cos n\theta \, d\theta + \int_{\phi}^{\pi} -\frac{h}{F} \cos n\theta \, d\theta \right\} = \frac{2 \sin n\phi}{n\pi}\eta$$

$A_1 = 2\sin\phi/\pi\ \eta,\ \ A_2 = \sin 2\phi/\pi\ \eta$

$$k = 2U\alpha \frac{1+\cos\theta}{\sin\theta} + 2U\left[\left(1-\frac{\phi}{\pi}\right)\frac{1+\cos\theta}{\sin\theta} + \Sigma_1^{\infty} \frac{2\sin n\phi}{n\pi} \sin n\theta\right] \quad (13.1)$$

$$C_L = 2\pi\alpha + 2\pi\eta\left(1 - \frac{\varphi}{\pi}\right) + 2\eta\sin\varphi$$

$$C_L = 2\pi\alpha + 2(\pi - \varphi + \sin\varphi)\eta \quad (13.2)$$

$$-C_{\mathrm{MLE}}=\pi/2\alpha+\pi/2\left[\eta\left(1-\frac{\varphi}{\pi}\right)+\frac{2\sin\varphi}{\pi}\eta-\frac{\sin2\varphi}{2\pi}\eta\right]$$

$$C_{\mathrm{MLE}}=-\pi/2\alpha+1/2\left[\pi-\varphi+\sin\varphi\left(2-\cos\varphi\right)\right]\eta$$

$$\mathrm{H}=-\int_{\mathrm{hinge}}^{\mathrm{trailingadge}}\mathrm{p}x'\mathrm{d}x$$

$$x' = c/2\,(1-\cos\theta) - c/2\,(1-\cos\varphi) = c/2\,(\cos\varphi - \cos\theta)$$

$$\mathrm{H}=-\int_{\varphi}^{\pi}2\rho U^2\left[\left\{\alpha+\eta\left(1-\frac{\varphi}{\pi}\right)\right\}\frac{(1+\cos\theta)}{\sin\theta}+\eta\sum_{1}^{\infty}\frac{2\sin n\varphi}{n\pi}\sin n\theta\right]\frac{c}{2}(\cos\varphi-\cos\theta)\frac{c}{2}\sin\theta\mathrm{d}\theta$$

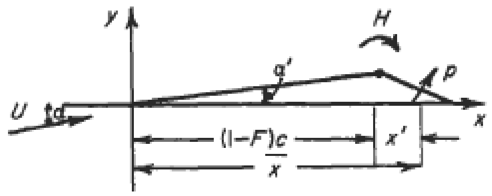

Fig. 13.3 Aerodynamic moment about the hinge line

$$-C_H F^2=\alpha\int_{\varphi}^{\pi}(1+\cos\theta)(\cos\varphi-\cos\theta)\,\mathrm{d}\theta+\eta\left\{\left(1-\frac{\theta}{\pi}\right)\cos\varphi I_1-\left(1-\frac{\varphi}{\pi}\right)I_2\right\}$$

$$\left(+\sum_{1}^{\infty}\frac{2\sin n\varphi}{n\pi}\cos\varphi I_3+\sum_{1}^{\infty}\frac{2\sin n\varphi}{n\pi}I_4\right)\quad(13.3)$$

$$I_1=\int_{\varphi}^{\pi}(1+\cos\theta)\mathrm{d}\theta=\pi-\varphi-\sin\varphi$$

$$I_2=\int_{\varphi}^{\pi}(1+\cos\theta)\cos\theta\mathrm{d}\theta=\left[\frac{\pi-\varphi}{2}\sin\varphi-\frac{\sin2\varphi}{4}\right]$$

$$I_3=\int_{\varphi}^{\pi}\sin n\theta\sin\theta\mathrm{d}\theta=1/2\left[\frac{\sin(n+1)\varphi}{n+1}-\frac{\sin(n-2)\varphi}{n-1}\right]$$

$$I_4=\int_{\varphi}^{\pi}\sin n\theta\sin\theta\cos\theta\mathrm{d}\theta=1/2\left[\frac{\sin(n+2)\varphi}{n+2}-\frac{\sin(n-2)\varphi}{n-2}\right]$$

$b_1 = \partial C_H/\partial\alpha$ and $b_2 = \partial C_H/\partial\eta$ $\qquad$ $b_1 = \partial C_H/\partial\alpha$ i $b_2 = \partial C_H/\partial\eta$

$$b_1=-\frac{1}{F^2}\int_{\varphi}^{\pi}(1+\cos\theta)(\cos\varphi-\cos\theta)\,d\theta$$

$$b_1=-\frac{1}{4F^2}\left\{2(\pi-\varphi)(2\cos\varphi-1)+4\sin\varphi-\sin2\varphi\right\}$$

$$b_2=\frac{\partial C_H}{\partial\eta}=\frac{1}{F^2}\times\text{coeficient of }\eta$$

$$b_2=-\frac{1}{4\pi F^2}\left\{(1-\cos2\varphi)-2(\pi-\varphi)^2(1-2\cos\varphi)+4(\pi-\varphi)\sin\varphi\right\}$$

$a_2 = 2(\pi - \phi + \sin\phi)$

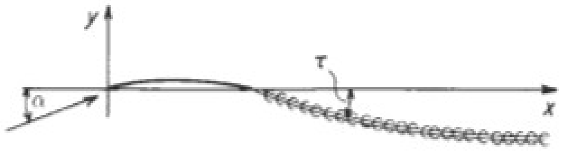

Fig. 13.4 Jet flap

$C_L = 4\pi A_0 \tau + 2\pi(1 + 2B_0)\alpha$

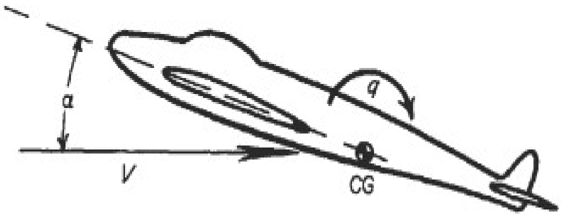

Fig. 13.5 Aeroplane is rotating with pitch velocity *q*

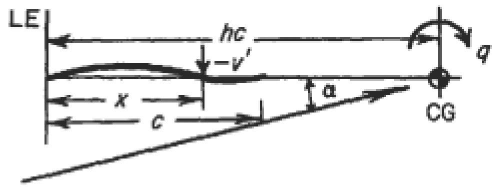

Fig. 13.6 Replacing the wing by the equivalent thin aerofoil

$v' = U\, dy_c/dx - U\alpha$ at $y = 0,\ 0 \le x \le 1$

becomes

$v' = U\, dy_c/dx - U\alpha$ при $y = 0,\ 0 \le x \le 1$

стає

$V\,(dy/dx - \alpha) = v - v'$

$dy/dx - \alpha = v/V - q/V\,(hc - x)$

$$\frac{dy_c}{dx} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} - \alpha = \mathrm{A}_0 - \alpha + \sum A_n \cos \mathrm{n}\theta \quad (13.4)$$

$$\frac{v}{V} = \mathrm{B}_0 + \sum B_n \cos \mathrm{n}\theta$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} - \alpha = \mathrm{B}_0 + \sum B_n \cos \mathrm{n}\theta - \frac{\mathrm{qc}}{V}\left(h - \frac{1}{2} + \frac{\cos\theta}{2}\right) \quad (13.5)$$

$$B_0 = \mathrm{A}_0 - \alpha - \frac{\mathrm{qc}}{V}\left(\frac{1}{2} - h\right), \mathrm{B}_1 = \mathrm{A}_1 + \frac{\mathrm{qc}}{V}, \mathrm{B}_n = \mathrm{A}_n.$$

$$k(\theta) = 2U\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

$$\mathrm{k}=2V\left[-B_0\left(\frac{1+\cos\theta}{\sin\theta}\right)+\sum B_n \sin \mathrm{n}\theta\right]$$

$C_L = (C_{L0}) + dC_L/d\alpha\ \alpha = \pi(A_1 - 2A_0) + 2\pi\alpha$

$$C_L=2\pi(\alpha - B_0)+\pi B_1=2\pi\left[\alpha - A_0+\frac{A_1}{2}+\left(\frac{3}{4}-h\right)\frac{\mathrm{qc}}{v}\right]$$

$$C_L=2a\left[\alpha - A_0+\frac{A_1}{2}+\left(\frac{3}{4}-h\right)\frac{\mathrm{qc}}{v}\right] \quad (13.6)$$

$C_{MLE} = -\ \pi/2\ [(\alpha - A_0) + A_1 - A_2/2]$

$$C_{\mathrm{MLE}}=\frac{\pi}{4}(B_2-B_1)-\frac{C_L}{4}=\frac{\pi}{4}(A_2-A_1)-\frac{\pi\,\mathrm{qC}}{8V}-\frac{1}{4}C_L$$

$$C_{\mathrm{MLE}}=\frac{\pi}{4}(A_2-A_1)-\frac{\pi}{8}\frac{c}{V}q-\frac{a}{4}\left\{\alpha - A_0+\frac{A_1}{2}+\left(\frac{3}{4}-h\right)\frac{\mathrm{qc}}{v}\right\}$$

$$C_{M\infty}=\mathrm{C_{MLE}}+\mathrm{hC}_L$$

$$C_{M\infty}=\frac{\pi}{4}(A_2-A_1)-\frac{2\pi}{16}\frac{\mathrm{qc}}{V}+\left(h-\frac{1}{4}\right)a\left[\alpha - A_0+\frac{A_1}{2}+\left(\frac{3}{4}-h\right)\frac{\mathrm{qc}}{v}\right]$$

$$C_{M\infty}=\mathrm{f}(A_n)-\left[\frac{a}{4}(1-2h)^2+\frac{2\pi-a}{16}\right]\frac{\mathrm{qc}}{v} \quad (13.7)$$

$$Z_q=\frac{\partial Z}{\partial q}=-\frac{\partial L}{\partial q}=-\frac{\partial C_L}{\partial q}\frac{1}{2}\rho \mathrm{V}^2\mathrm{S}=-\frac{1}{2}\rho\mathrm{V}^2\mathrm{Sa}\left(\frac{3}{4}-h\right)\frac{c}{V}$$

$$z_q=-\frac{a}{2}\left(\frac{3}{4}-h\right)\frac{c}{l_t}$$

$$Z_q=-\rho\mathrm{V}\int_{-s}^{s}\frac{a}{2}\left(\frac{3}{4}-h\right)c^2\,\mathrm{dy}$$

$$z_q=\frac{-1}{\mathrm{Sl}_t}\int_{-s}^{s}\frac{a}{2}\left(\frac{3}{4}-h\right)c^2\,\mathrm{dy}$$

$$M_q=\frac{\partial M_{\mathrm{CG}}}{\partial q}=\frac{\partial C_{\mathrm{MCG}}}{\partial q}\frac{1}{2}\rho\mathrm{V}^2\mathrm{Sc}=-\frac{1}{2}\rho\mathrm{V}^2\mathrm{Sc}\left\{\frac{a}{4}(1-2h)^2+\frac{2\pi-a}{16}\right\}\frac{c}{V}$$

$$-\left[\frac{a}{8}(1-2h)^2+\frac{2\pi-a}{16}\right]\mathrm{VSc}^2$$

$$m_q=\frac{M_q}{\rho\,\mathrm{VSl}_t^2}=-\left[\frac{a}{8}(1-2h)^2+\frac{2\pi-a}{32}\right]\frac{c^2}{l_t^2}$$

$$M_q = -\rho V \int_{-s}^{s} \left[\frac{a}{8}(1-2h)^2 + \frac{2\pi - a}{32} \right] c^3 \, \mathrm{dy}$$

$$m_q = -\frac{1}{\mathrm{Sl}_t^2} \int_{-s}^{s} \left[\frac{a}{8}(1-2h)^2 + \frac{2\pi - a}{32} \right] c^3 \, \mathrm{dy}$$

$p_1 - p_2 = ½ \rho (u^2_2 - u^2_1) = \rho (u_2 - u_1) (u_2 + u_1)/2$

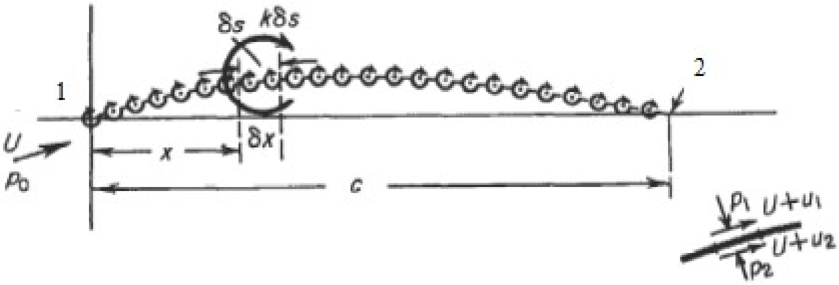

Fig. 13.7 Insert shows velocity and pressure above and below δs
1 – leading edge; 2 – trailing edge

$$(p_1 - p_2)\,\delta s = \rho U k \,\delta s$$

$$v = \int_0^c \frac{k}{2\pi} \frac{\mathrm{dx}}{x - x_1}$$

$$U\left[\frac{\mathrm{dy}_c}{\mathrm{dx}} - \alpha\right] = \int_0^c \frac{k}{2\pi} \frac{\mathrm{dx}}{x - x_1}$$

$$\delta m = \sigma(x_1)\,\delta x_1$$

δm = outflow across boundary ($y_t \pm dy_t/dx_1 \;\; \delta x_1$) - inflow across $\pm y_t$

$=2\,[(U + u' - du'/dx_1\, \delta x_1)(y_t + dy_t/dx_1\, \delta x_1) - (U + u')y_t]$

$$\delta m = 2U \frac{\mathrm{dy}_t}{\mathrm{dx}_1} \delta x_1$$

($\delta\phi = \delta m/2\pi \ln r = U/\pi \, dy_t/dx_1 \, \delta x_1 \ln r$

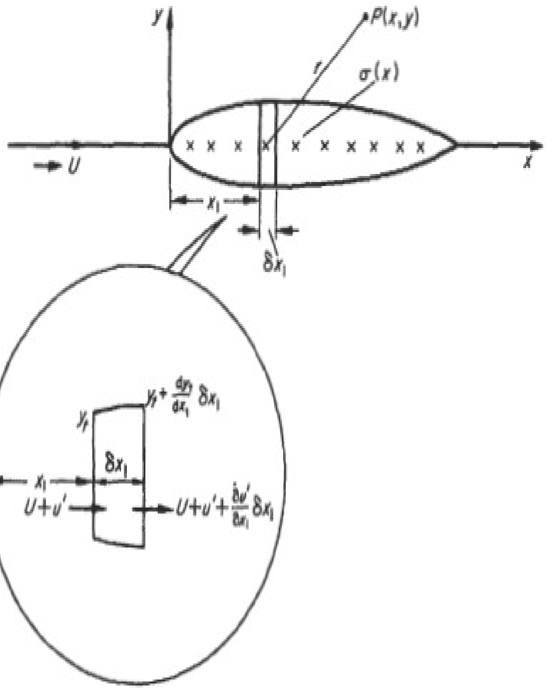

Fig. 13.8 Distribution of sources, and sinks

$$\phi = \frac{U}{\pi}\int_0^c \frac{dy_t}{dx_1}\ln r dx_1$$

$$\phi = Ux + \frac{U}{\pi}\int_0^c \frac{dy_t}{dx_1}\ln r dx_1$$

$$u = \frac{\partial\phi}{\partial x} = U + \frac{U}{\pi}\int_0^c \frac{dy_t}{dx_1}\frac{(x-x_1)}{(x-x_1)^2+y^2}dx_1$$

$$u = \frac{\partial\phi}{\partial y} = U + \frac{U}{\pi}\int_0^c \frac{dy_t}{dx_1}\frac{y}{(x-x_1)^2+y^2}dx_1$$

$$u = U + u' = U + \frac{U}{\pi}\int_0^c \frac{dy_t}{dx_1}\frac{1}{x-x_1}dx_1$$

$$C_p = -2\frac{u'}{U} = -\frac{2}{\pi}\int_0^c \frac{dy_t}{dx_1}\frac{1}{x-x_1}dx_1$$

Control questions
1. What is the distribution of contributions to total lift of cambered flapped aerofoil?
2. What is the distribution of chordwise circulation due to flap deflection?
3. Describe the aerodynamic moment about the hinge line.
4. What is the assessment of all parameters of flapped aerofoil gives the thin aerofoil theory?
5. How does a jet flap contribute to lift?
6. How to determine the lift coefficient of a pitching rectangular wing and pitching-moment coefficient about the leading edge?
7. How is the non-dimensional normal force derivative due to pitching z_q and the non-

dimensional pitching moment derivative due to pitching m_q determined?.
8. How is the thickness problem solved for the thin-aerofoil theory?
9. How to solve the thickness problem for thin aerofoils?

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Lecture 14. The vortex systems

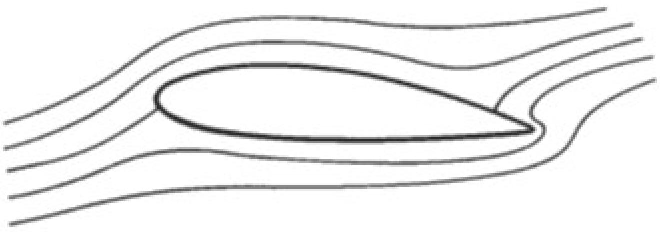

Fig. 14.1 Streamlines of the flow around an aerofoil with zero circulation, stagnation point on the rear upper surface

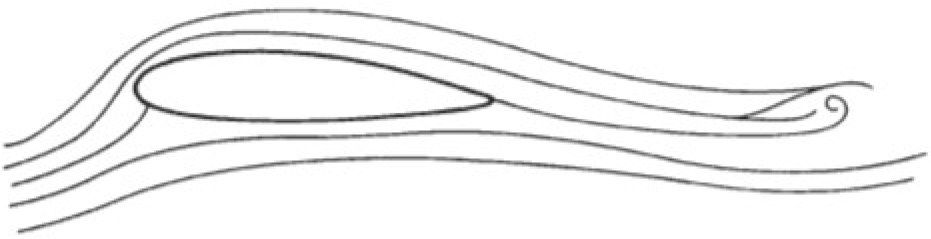

Fig. 14.2 Streamlines of the flow around an aerofoil with full circulation, stagnation point at the trailing edge. The initial eddy is left way behind

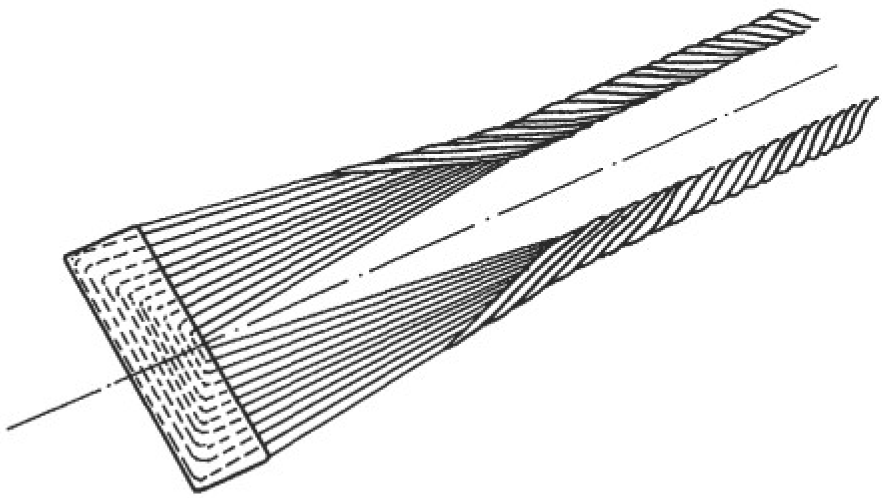

Fig. 14.3 The horseshoe vortex

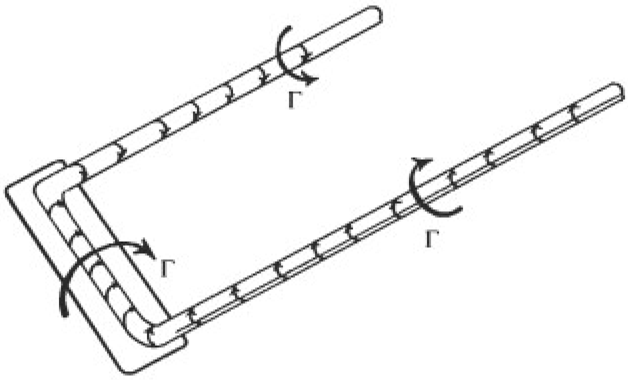

Fig. 14.4 The simplified horseshoe vortex

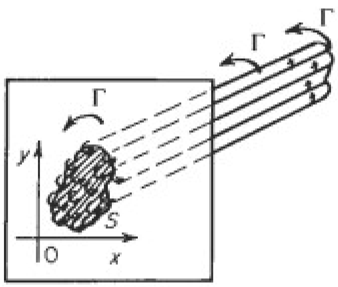

Fig. 14.5 The vorticity of a section of vortex tube

$\Gamma = \xi S$

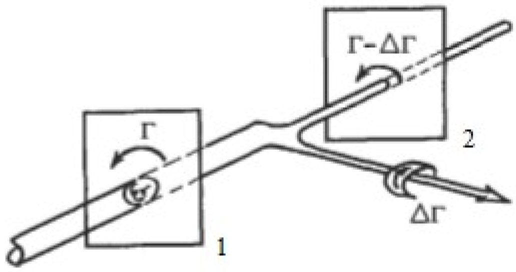

Fig. 14.6

1 – section A; 2 – section B

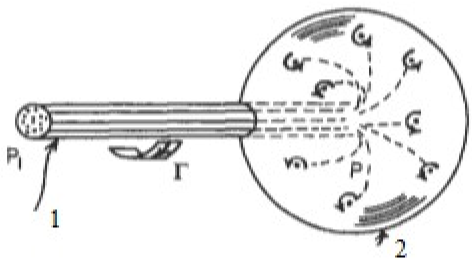

Fig. 14.7

1 – vortex tube strength Γ; 2 - Spherical boundary surrounding ‘free’ end at point P

$\Gamma' = 2\pi R \sin\theta v$ (14.1)

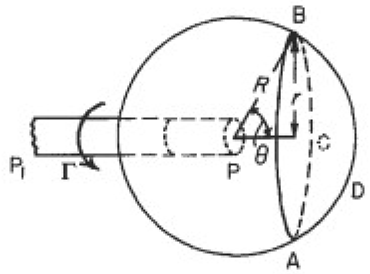

Fig. 14.8

$\Gamma' = 2\pi r v$ (14.2)

Γ' = (surface area of cap)/(surface area of sphere) Γ

$= [2\pi R^2(1-\cos\theta)]/4\pi R^2\ \Gamma = \Gamma/2\ (1-\cos\theta)$ (14.3)

$v = \Gamma/4\pi r\ (1-\cos\theta)$ (14.4)

$v_1 = -\Gamma/4\pi r\ (1-\cos\theta_1)$ (14.5)

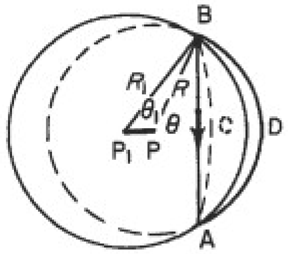

Fig. 14.9

$v-v_1 = \Gamma/4\pi r\ [1-\cos\theta - (1-\cos\theta_1)] = \Gamma/4\pi r\ (\cos\theta_1-\cos\theta)$

$\cos\theta_1 \rightarrow \cos(\theta - \delta\theta) = \cos\theta + \sin\theta\ \delta\theta$

$v-v_1 \rightarrow \delta v$

$\delta v = \Gamma/4\pi r \sin\theta\ \delta\theta$

$\delta v = \Gamma/4\pi R^2 \sin\theta\ \delta s$ (14.7)

$\delta v = \Gamma/4\pi r^2 \sin\theta\ \delta s$ (14.8)

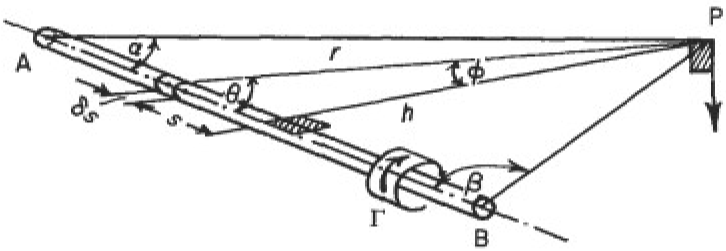

Fig. 14.10

$\phi_A = -(\pi/2 - \alpha)$ to $\phi_B = +(\pi/2 - \beta)$

$\sin\theta = \cos\phi$, $r^2 = h^2 \sec^2\phi$

$ds = d(h \tan\phi) = h \sec^2\phi \, d\phi$

$$v = \int_{-\left(\frac{\pi}{2}-\alpha\right)}^{+\left(\frac{\pi}{2}-\beta\right)} \frac{\Gamma}{4\pi h}\cos\phi d\phi = \frac{\Gamma}{4\pi h}\left[\sin\left(\frac{\pi}{2}-\beta\right)+\sin\left(\frac{\pi}{2}-\alpha\right)\right] = \frac{\Gamma}{4\pi h}(\cos\alpha - \cos\beta) \quad (14.9)$$

$v = \Gamma/4\pi h\,(\cos\alpha + 1)$ (14.10)

$v = \Gamma/4\pi h$ (14.11)

$v = \Gamma/2\pi h$ (14.12)

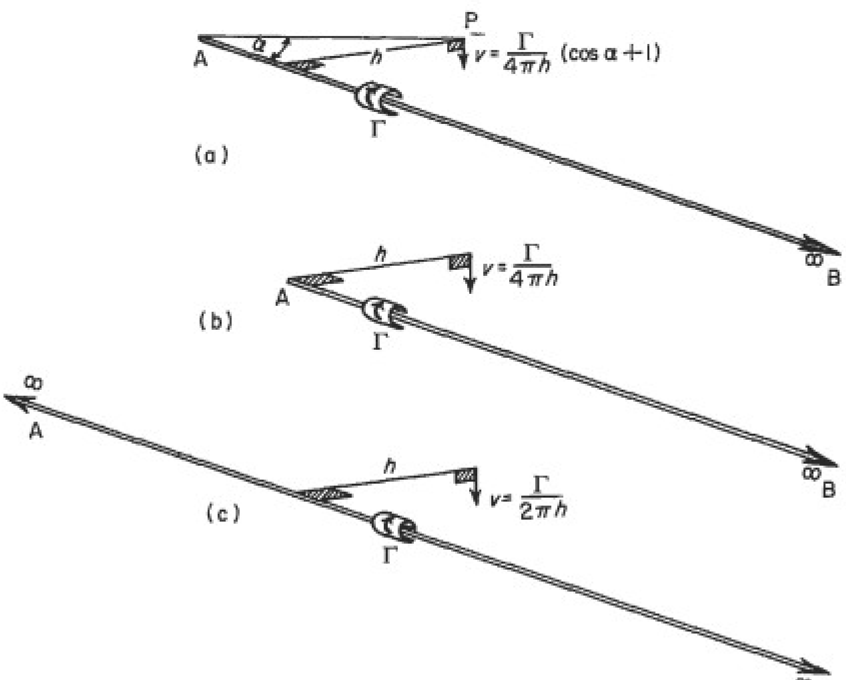

Fig. 14.11

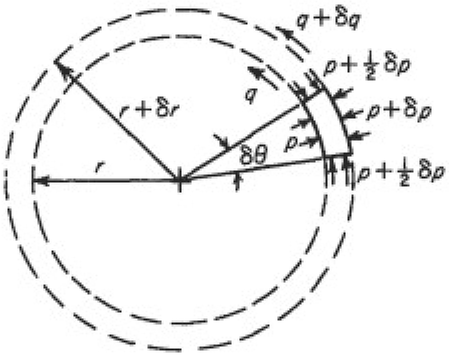

Fig. 14.12 Motion of an element outside a vortex core

$(p+\delta p)(r+\delta r)\ \delta\theta - pr\delta\theta - 2(p+1/2\delta p)\ \delta r\ 1/2\ \delta\theta$

mass $(\text{velocity})^2/\text{radius} =$

$= pr\delta r\ \delta\theta\ q^2/r = pq^2\ \delta r\ \delta\theta$

$r\ \delta p = pq^2\ \delta r$ since $\delta\theta \neq 0$ (14.13)

$p + \frac{1}{2}\rho q^2 = (p+\delta p) + \frac{1}{2}\rho\ (q+\delta q)^2$

$\delta p + \rho q\delta q = 0$

$\delta p = -\ \rho q\delta q$ (14.14)

$\rho\ q^2\ \delta r + \rho qr\ \delta q = 0$

$q\ \delta r + r\ \delta q = 0$

$\delta\ (qr) = 0$

$qr = \text{constant}$ (14.15)

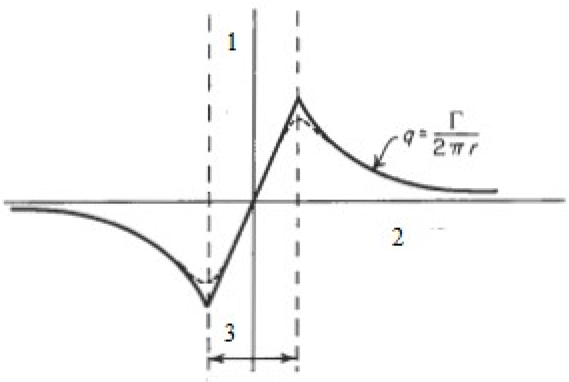

Fig. 14.13 Velocity distribution in a real vortex with a core
1 – velocity q; 2 – radius r; 3 - The core

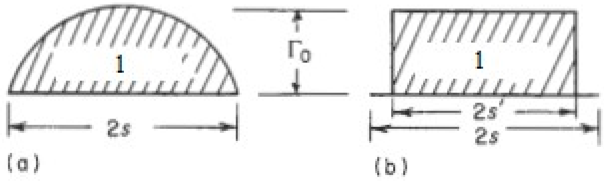

Fig. 14.14
1 - Total lift ρV; (a) Normal loading (b) Equivalent simplified loading

s'/s – (total lift)$/2s\rho V\Gamma_0$ (14.16)

$L = \rho V^2 s^2 2\pi A_1$

$\Gamma_0 = 4sV\Sigma A_n \sin n\,\pi/2$
$s'/s = (\rho V^2 s^2 2\pi A_1)/(2\rho\, V^2 4s^2\, \Sigma A_n \sin n\,\pi/2) = \pi/4\ A_1/(A_1 + A_3 + A_5 + A_7 \ldots)$

$s/s' = 4/\pi\,(1 - A_3/A_1 + A_5/A_1 + A_7/A_1 \ldots)$ (14.17)

$A_3 = A_5 - A_7 = 0$
$s' = (\pi/4)s$ (14.18)

Control questions
1. What is the practical application of the Lanchester-Prandtl theory in the design of aircraft?
2. What is the theoretical model that Lanchester proposed to replace the wing?
3. What are the three main parts of the vortex system equivalent to the generated lift?
4. What is it the initial vortex?
5. What is it the final vortex system?
6. What is it a horseshoe vortex?
7. What is it a connected vortex system?
8. What is the essence of the four fundamental theorems of vortex motion in the inviscid Helmholtz flow?
9. Describe the law of Bio-Savar?
10. What are the special cases of the law of Bio-Savar?
11. How is the change in velocity in the vortex flow?
12. Give a description of a simplified horseshoe vortex?

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Lecture 15. Flow field modelling with vortex elements

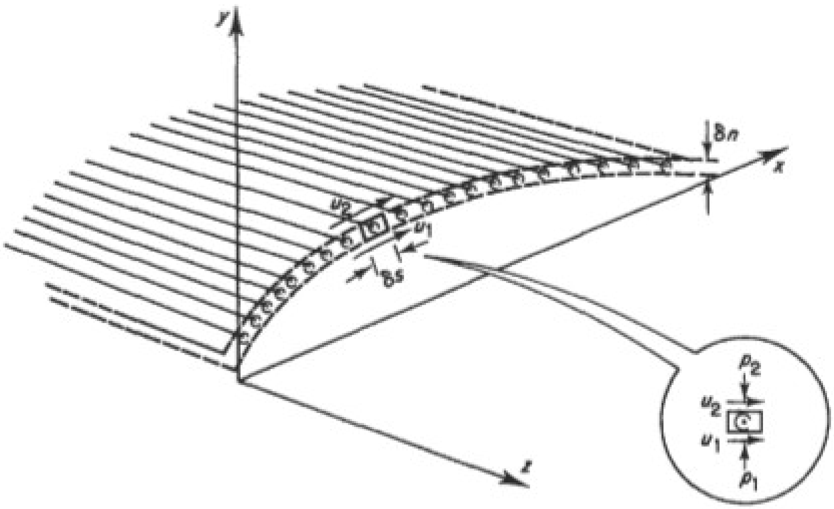

Fig. 15.1 Sheet of vortex filaments

$\Delta\Gamma = k\delta s$ (15.1)

$\Delta\Gamma = (u_2 - u_1)\,\delta s$ (15.2)

$$C_p = \frac{-u'}{U} = \frac{1}{\pi}\underbrace{\int_{-s}^{s}\int_{x_1(z)}^{x_1(z)-c(z)}}_{l_1}\frac{\mathrm{dy}_t}{\mathrm{dx}}(\mathrm{x,z})\frac{x - x_1}{\left[\left(x - x_1\right)^2 + \left(z - z_1\right)^2\right]^{3/2}}\,\mathrm{dxdy} \qquad (15.3)$$

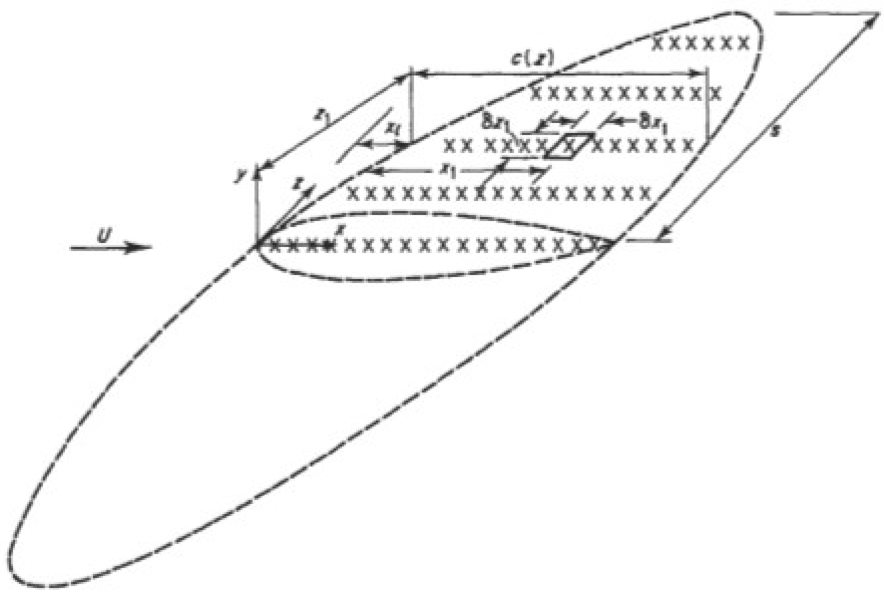

Fig. 15.2 Modelling the displacement effect by a distribution of sources

$$I_1 = \frac{1}{c}\int_0^1 S_t(x)\, x \underbrace{\int_{-(s+z_1)/s}^{(s-z_1)/s} \frac{dz}{\left(x^2+z^2\right)^{3/2}}}_{I_1} dx$$

$$I_2 = -\int_{\frac{-c}{(s-z_1)}}^{-\infty} \frac{\chi}{\left(z^2\chi^2+1\right)^{\frac{3}{2}}} d\chi - \int_{\infty}^{\frac{c}{(s-z_1)}} \frac{\chi}{\left(z^2\chi^2+1\right)^{\frac{3}{2}}} d\chi$$

$$\frac{1}{z^2}\left[\frac{-1}{\sqrt{\left(\frac{z_1 c}{s+z}\right)^2+1}} - \frac{1}{\sqrt{\left(\frac{z_1 c}{s-z}\right)^2+1}}\right]$$

$$\left(\frac{z_1 c}{s+z}\right)^2 \ll 1 \wedge \left(\frac{z_1 c}{s-z}\right)^2 \ll 1$$

$I_2 \cong -2/z_1^2$

$$C_p \simeq -\frac{2}{\pi}\int_{x_l}^{c+x_l} \frac{dy_t}{dx}\frac{1}{x-x_1} dx$$

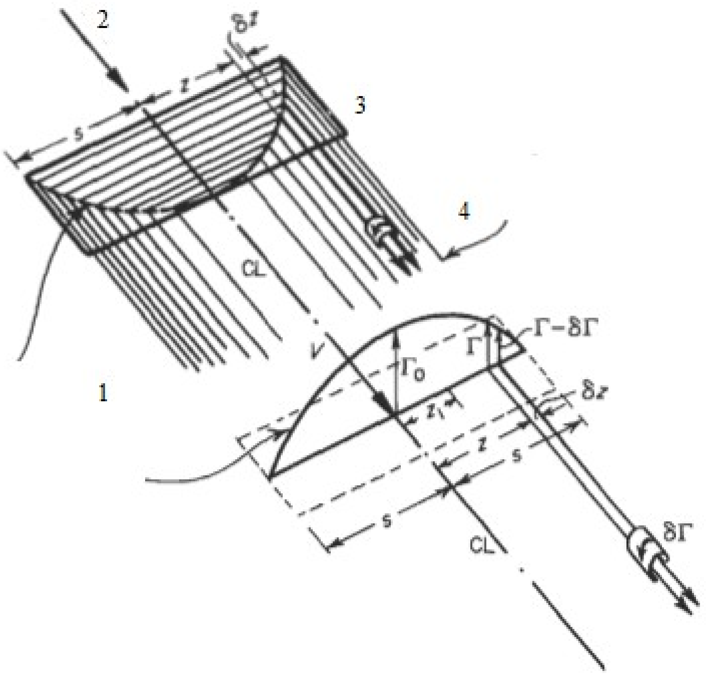

Fig. 15.3 The relation between spanwise load variation and trailing vortex strength
1 – curve defining the spanwise variant in strength of the combined bound vortex filaments; 2 - forward velocity V; 3 – aerofoil with hypothetical spanwise bound vortex filaments; 4 – trailing vortex filaments

$\delta\Gamma = df(z)/dz\ \delta z$ (15.25)

$l = \rho V \Gamma$

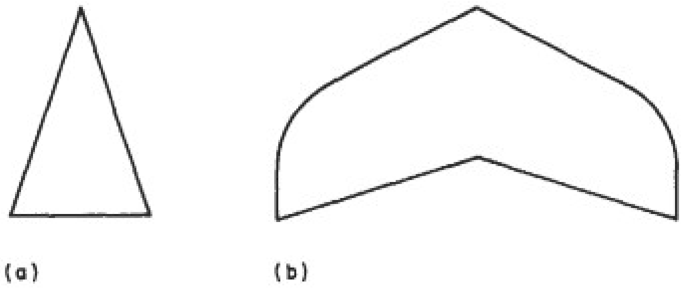

Fig. 15.4 wing shapes for which vortex-sheet model is not suitable
(a) Delta wing; (b) Swept - back wing

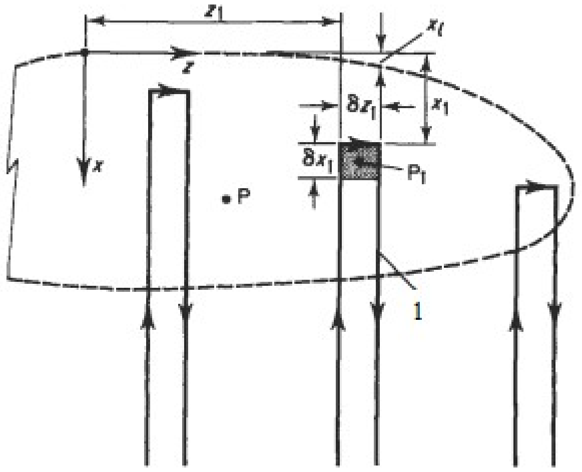

Fig. 15.5 Modelling the lifting effect by a distribution of horseshoe vortex elements:
1 – vortex strength, $k\delta x_1$

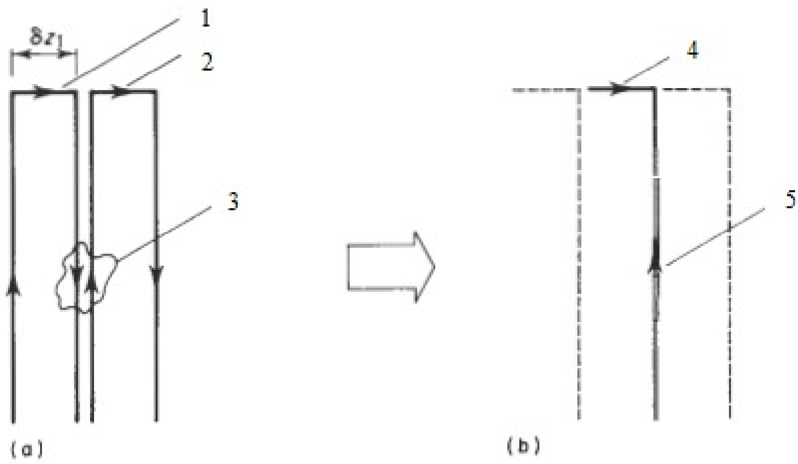

Fig. 15.6 Equivalence between distributions of (a) horseshoe and (b) L-shaped vortices
1 – strength $k\delta x_1$; 2 – strength $(k + \partial k/\partial z_1\ \delta z_1)\delta x_1$; 3 – partial cancellation; 4 – strength $k\delta x_1$; 5 – strength $\partial k/\partial z_1\ \delta z_1 \delta x_1$

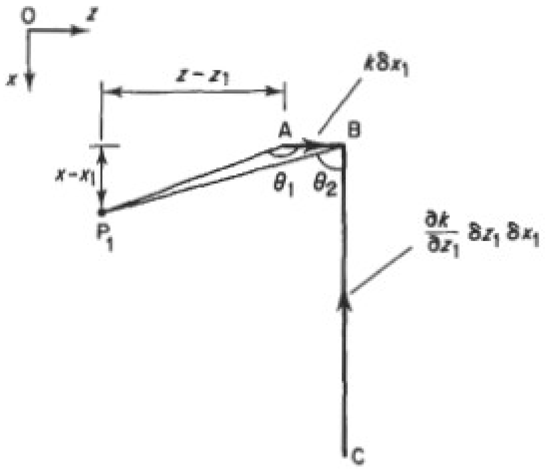

Fig. 15.7 Geometric notation for L-shaped vortex element

$$\delta v_i(x_1, z_1) = (\delta v_i)_{AB} + (\delta v_i)_{BC}$$

$$-\frac{k\delta x}{4\pi(x-x_1)}\left[\cos\theta_1 - \cos\left(\theta_2 + \frac{\pi}{2}\right)\right] + \frac{1}{4\pi}\frac{\partial k}{\partial z}\delta z\delta x\frac{1+\cos\theta_2}{z+\delta z - z_1} \quad (15.26)$$

$$\cos\theta_1 = \frac{z - z_1}{\sqrt{(x-x_1)^2 + (z-z_1)^2}}$$

$$\cos\theta_2 = \frac{-x - x_1}{\sqrt{(x-x_1)^2 + (z+\delta z - z_1)^2}}$$

$$\cos\left(\theta_2 + \frac{\pi}{2}\right) = -\sin\theta_2 = \frac{z + \delta z - z_1}{\sqrt{(x-x_1)^2 + (z+\delta z - z_1)^2}}$$

$(a + b)^n = a^n + nd^{n-1} b + ..;$

$[(x - x_1)^2 + (z + \delta z - z_1)^2]^{-1/2} = 1/r - (z - z_1)/r^2\ \delta z + \ldots$

$$\cos\theta_1 = \frac{z-z_1}{r} \quad (15.27)$$

$$\cos\theta_2 = \frac{-x-x_1}{r} + \frac{(x-x_1)(z-z_1)}{r^3}\delta z + \ldots \quad (15.28)$$

$$\cos\left(\theta_2 + \frac{\pi}{2}\right) = \frac{z-z_1}{r} + \left[\frac{1}{r} - \frac{(z-z_1)^2}{r^3}\right]\delta z + \ldots \quad (15.29)$$

$$\delta v_i = \frac{k}{4\pi}\delta x\delta z\frac{(x-x_1)}{r^3} + \frac{1}{4\pi}\frac{\partial k}{\partial z}\delta x\delta z\left[\frac{1}{z-z_1} - \frac{x-x_1}{r(z-z_1)}\right] \quad (15.30)$$

$$v_i(x_1, z_1) = \frac{1}{4\pi}\int_{-s}^{s}\int_{x_l}^{x_l+c}\left\{\frac{\partial k}{\partial z}\left[\underbrace{\frac{1}{z-z_1}}_{(a)} - \underbrace{\frac{x-x_1}{r(z-z_1)}}_{(b)}\right] + k\underbrace{\frac{x-x_1}{r^3}}_{(c)}\right\}dxdz \quad (15.31)$$

$$v_i(z_1)=\mathrm{w}(z_1)=\frac{1}{4\pi}\int_{-s}^{s}\frac{\mathrm{d}\Gamma}{\mathrm{dz}}\frac{1}{z-z_1}\mathrm{dz} \qquad (15.32)$$

$$\Gamma(z)=\int_{x_l}^{c(z)+x_l} k(\mathrm{x,z})\,\mathrm{dx} \quad (15.33)$$

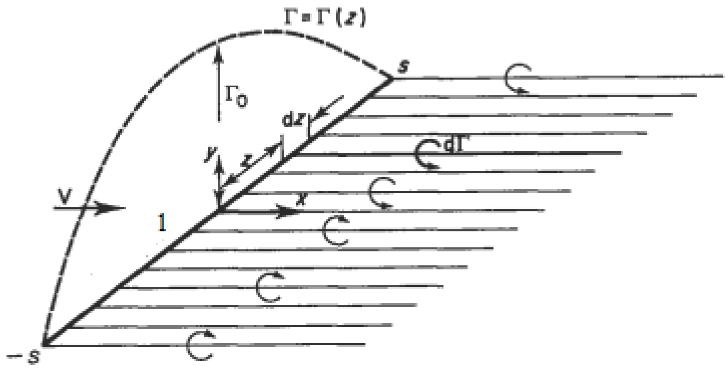

Fig. 15.8 Prandtl's lifting line (1) model

$$\delta \mathrm{v}_i(z_1)=\frac{1}{4\pi(z-z_1)}\frac{\mathrm{df}}{\mathrm{dz}}\delta \mathrm{z}$$

$$\varepsilon=\tan^{-1}\frac{w}{V}\simeq\frac{w}{V}$$

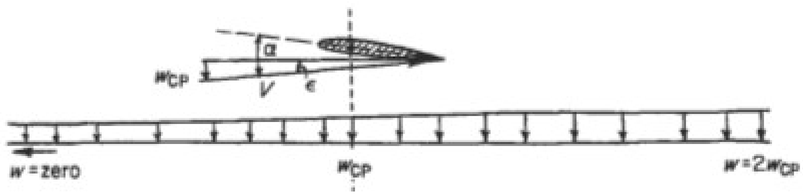

Fig. 15.9 Variation in magnitude of downwash in front of and behind wing

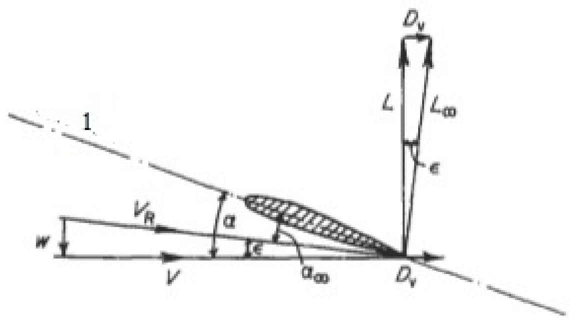

Fig. 15.10 The influence of downwash on wing velocities and forces: w = downwash; V = forward speed of wing; V_R = resultant oncoming flow at wing; α = incidence; ε = downwash angle = w/V; α_∞ = (α - ε) = equivalent two-dimensional incidence; L_∞ = two-dimensional lift; L = wing lift; D_v = trailing vortex drag; 1 – chord line

$l = \rho V \Gamma$

$$L=\int_{-s}^{s}\rho V\Gamma dz \quad (15.34)$$

$$d_v = \rho w\Gamma \quad (15.35)$$

$$D_v=\int_{-s}^{s}\rho w\Gamma dz \quad (15.36)$$

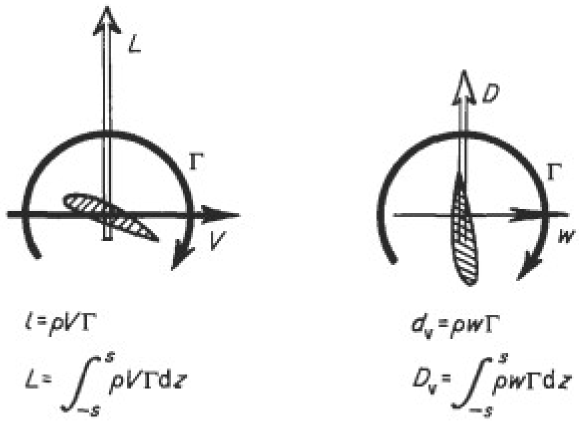

Fig. 15.11 Circulation superimposed on forward wind velocity and downwash to give lift and vortex drag (induced drag) respectively

Control questions
1. Describe the vortex veil and its use to model the lift of the wing.
2. Give a description of modeling the impact of substitution by source distribution.
3. Give a description of the model of the lifting force of the vortex veil.
4. Describe the relationship between the transverse load and the final vortex.
5. Give a description of the model of the Prandtl lifting line.
6. Give a description of the downstream slope and the resistance of the final vortex.
7. Give a description of the induced velocity.

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Lecture 16. Finite wing theory

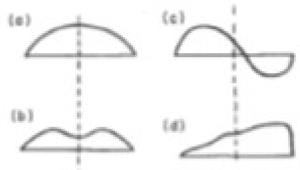

Fig. 16.1 Typical spanwise distributions of lift:
a) Isolated wing in steady symmetric flight; b) Lift distribution modified by fuselage effect; c) Lift distribution in antisymmetric flight; d) Antisymmetric flight with ailerons in operation

$\Gamma = C_L/2\ Vc$

$\Gamma = 4C_\Gamma s$

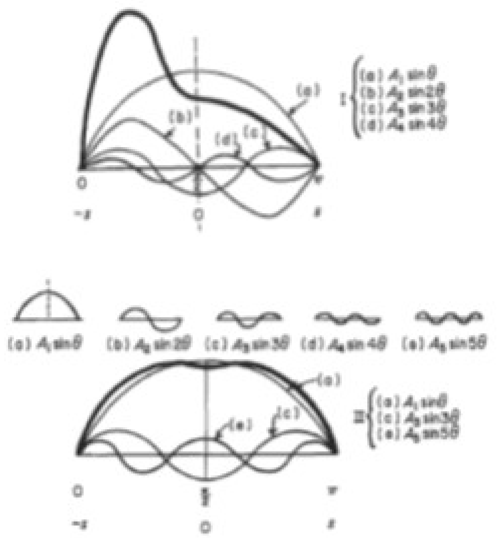

Fig. 16.2 Loading make-up by selected sine series

$$\Gamma = 4\,\mathrm{sV} \sum_i^{\infty} A_n \sin n\theta \quad (16.1)$$

$$l = 4\,\rho V^2 s \sum_i^{\infty} A_n \sin n\theta$$

$$L = \int_{-s}^{s} \rho V \Gamma dz$$

$$L = \int_{-s}^{s} \rho V \Gamma s \sin\theta d\theta$$

$$L = \int_0^{\pi} \rho V s^2 \sum A_n \sin n\theta \sin\theta d\theta = 4\,s^2 \rho V^2 \int_0^{\pi} \sum A_n \left[\cos(n-1)\theta - \cos(n+1)\theta\right] d\theta$$

$$4\,s^2 \rho V^2 \frac{1}{2}\left[\sum A_n \left(\frac{\cos(n-1)\theta}{(n-1)} - \frac{\cos(n+1)\theta}{(n+1)}\right)\right]_0^{\pi}$$

$$A_1 \frac{\sin(n-1)\theta}{(n-1)}\Big|_0^{\pi} = \mathrm{A}_1 \pi$$

L = A_1 π ½ ρ $V^2 4s^2$ = C_L ½ ρ V^2S

C_L = π A_1 (AR) (16.2)

C_L = π (AR) A_1 and L = 2π ρ $V^2 s^2$ A_1 (16.2a)

$$w_{\theta 1} = \frac{I}{4\,\pi \mathrm{s}} \int_0^{\pi} \frac{\frac{\mathrm{d}\Gamma}{\mathrm{d}\theta}\,\mathrm{d}\theta}{(\theta - \theta_1)}$$

dΓ/dθ = 4sV Σ n A_n cos nθ

$$w_{\theta 1} = \frac{4\,\mathrm{sV}}{4\,\pi \mathrm{s}} \int_0^{\pi} \frac{\Sigma \mathrm{nA}_n \cos \mathrm{n}\theta}{(\cos\theta - \cos\theta_1)}\,\mathrm{d}\theta = \frac{V}{\pi}\,\Sigma \mathrm{nA}_n G_n$$

$$\mathrm{w} = \mathrm{V}\frac{\Sigma \mathrm{nA}_n \sin \mathrm{n}\theta}{\sin\theta}\ (16.3)$$

$$D_v = \int_{-s}^{s} \rho \mathrm{w} \Gamma \mathrm{dz}$$

$$D_v = \int_0^{\pi} \rho \underbrace{\frac{\mathrm{V}\Sigma \mathrm{nA}_n \sin \mathrm{n}\theta}{\sin\theta}}_{w} \underbrace{4\,\mathrm{sV}\Sigma \mathrm{A}_n \sin \mathrm{n}\theta}_{\Gamma} \underbrace{\mathrm{s}\sin\theta \mathrm{d}\theta}_{\mathrm{dz}}$$

$$\rho \mathrm{V}^2 s^2 \int_0^{\pi} \Sigma \mathrm{nA}_n \sin\theta \Sigma \mathrm{A}_n \sin \mathrm{n}\theta$$

$$\frac{\pi}{2} \Sigma \mathrm{nA}_n^2$$

$$\mathrm{I} = \int_0^{\pi} (A_1 \sin\theta + 3\,A_3 \sin 3\theta + 5\,A_5 \sin 5\theta)(A_1 \sin\theta + \mathrm{A}_3 \sin 3\,\theta + \mathrm{A}_5 \sin 5\,\theta)$$

$$\int_0^{\pi} \{A_1^2 \sin^2\theta\} + 3\,A_3^2 \sin^2\theta + 5\,A_5^2 \sin^2\theta + [A_1 A_3 \sin\theta \sin 3\theta]$$

$$\mathrm{I} = \int_0^{\pi} (A_1^2 \sin^2\theta + 3\,A_3^2 \sin^2 3\theta + 5\,A_5^2 \sin^2 5\,\theta + \ldots)\,\mathrm{d}\theta = \frac{\pi}{2}[A_1^2 + 3\,A_3^2 + 5\,A_5^2 + \ldots] = \frac{\pi}{2}\Sigma \mathrm{nA}_n^2$$

$$D_v = 4\,\rho \mathrm{V}^2 s^2 \frac{\pi}{2} \Sigma \mathrm{nA}_n^2 = \mathrm{C}_{\mathrm{Dv}} \frac{1}{2} \rho \mathrm{V}^2 S$$

$$C_{\mathrm{Dv}} = \pi(\mathrm{AR})\Sigma \mathrm{nA}_n^2\ \ (16.4)$$

$$A_1^2 = \frac{C_L^2}{\pi^2 (\mathrm{AR})^2}$$

$$C_{\mathrm{Dv}} = \frac{C_L^2}{\pi(\mathrm{AR})} \Sigma \mathrm{n}\left(\frac{A_n}{A_1}\right)^2 = \frac{C_L^2}{\pi(\mathrm{AR})}\left[1 + \left(\frac{3\,A_3^2}{A_1^2} + \frac{5\,A_5^2}{A_1^2} + \frac{7\,A_7^2}{A_1^2} + \ldots\right)\right]$$

$$\left(\frac{3A_3^2}{A_1^2}+\frac{5A_5^2}{A_1^2}+\frac{7A_7^2}{A_1^2}+\ldots\right)$$

$$C_{\mathrm{Dv}}=\frac{C_L^2}{\pi(\mathrm{AR})}[1+\delta] \quad (16.5)$$

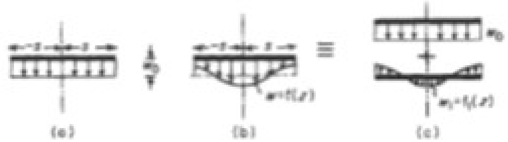

Fig. 16.3 (a) Elliptic distribution gives constant downwash and minimum drag. (b) Non-elliptic distribution gives varying downwash. (c) Equivalent variation for comparison purposes

$$L\propto\int_{-s}^{s}\dot{m}w_0\,\mathrm{dz}$$

$$L\propto\int_{-s}^{s}\dot{m}\left(w_0-f_1(z)\right)\mathrm{dz}$$

$$L\propto\int_{-s}^{s}\dot{m}f_1(z)\,\mathrm{dz} \quad (16.6)$$

$$D_{v(a)}\propto\frac{1}{2}\dot{m}\int_{-s}^{s}w_0^2\,\mathrm{dz} \quad (16.7)$$

$$D_{v(b)}\propto\frac{1}{2}\dot{m}\int_{-s}^{s}\left(w_0-f_1(z)\right)^2\mathrm{dz}\propto\frac{1}{2}\dot{m}\int_{-s}^{s}\left\{w_0^2+2w_0f_1(z)+\left[f_1(z)\right]^2\right\}\mathrm{dz}$$

and since $\int_{-s}^{s}\dot{m}f_1(z)=0$ in Eqn (16.6)

$$D_{v(b)}\propto\left[\frac{1}{2}\dot{m}\int_{-s}^{s}w_0^2\,\mathrm{dz}\right]+\frac{1}{2}\dot{m}\int_{-s}^{s}\left[f_1(z)\right]^2\mathrm{dz} \qquad (16.8)$$

$$D_{v(b)}=\mathrm{D}_{v(a)}+\frac{1}{2}\dot{m}\int_{-s}^{s}\left[f_1(z)\right]^2\mathrm{dz}$$

$$\int_{-s}^{s}\left[f_1(z)\right]^2\mathrm{dz}>0$$

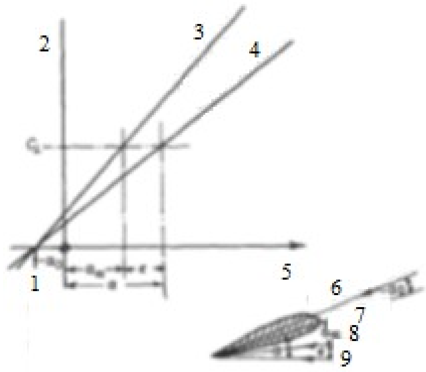

Fig. 16.4 Lift-versus-incidence curve for an aerofoil section of a certain profile, working two-dimensionally and working in a flow regime influenced by end effects, i.e. working at some point along the span of a finite lifting wing
1 – no lift incidence; 2 – section lift coefficient; 3 – two dimentional lift scope α_∞; 4 - three dimentional lift scope α; 5 - incidence; 6 – no lift wing direction; 7 – chord line; 8 – equivalent two dimentional wind V_∞; 9 – apparent wind V

$C_L = \alpha_\infty (\alpha_\infty - \alpha_0) = \alpha (\alpha - \alpha_0)$

$C_L = \alpha_\infty [(\alpha - \alpha_0) - \varepsilon]$ (16.9)

C_L = lift per unit span/

$(1/2\ \rho V^2 c) = l/(1/2\ \rho V^2 c) = \rho V\Gamma/(1/2\ \rho V^2 c)$

$C_L = 2\Gamma/Vc$ (16.10)

$2\Gamma/c\alpha_\infty = V[(\alpha - \alpha_0) - \varepsilon]$

$$V\varepsilon = w = \frac{1}{4\pi}\int_{-s}^{s}\frac{d\Gamma/dz}{z - z_1}dz$$

$$\frac{2\Gamma}{c\alpha_\infty} = V(\alpha - \alpha_0) + \frac{1}{4\pi}\int_{-s}^{s}\frac{d\Gamma/dz}{z - z_1}dz \quad (16.11)$$

$\Gamma = 4sV\Sigma A_n \sin n\theta$

$w = V\Sigma nA_n \sin n\theta/\sin\theta$

$2\ 4sV\Sigma A_n \sin n\theta/c\alpha_\infty = V(\alpha - \alpha_0) - V\Sigma nA_n \sin n\theta/\sin\theta$

$$\mu(\alpha - \alpha_0) = \sum_{i=1}^{\infty} A_n \sin n\theta\left(1 + \frac{\mu n}{\sin\theta}\right) \quad (16.12)$$

$$\Gamma = 4sVA_1\sqrt{1 - \left(\frac{z}{s}\right)^2}$$

$\mu(\alpha - \alpha_0) = A_1 \sin\theta\,(1 + \mu/\sin\theta)$

$A_1 = \mu/(\sin\theta + \mu)(\alpha - \alpha_0)$ (16.13)

$l = \rho V\Gamma = C_L \rho V^2 c$ (16.14)

$$c = c_0\sqrt{1 - \left(\frac{z}{s}\right)^2} = c_0 \sin\theta$$

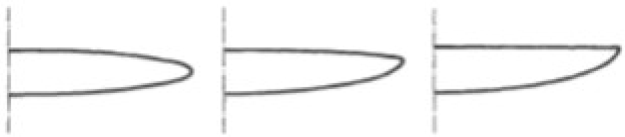

Fig. 16.5 Three different wing planforms with the same elliptic chord distribution

$\Gamma = C_L V/2\ c$

$C_L = a_\infty [(\alpha - \alpha_0) - \varepsilon]$ from Eqn (16.9)
$\Gamma \propto c a_\infty [(\alpha - \alpha_0) - \varepsilon]$

$\mu = \mu_0 \sin\theta$ where $\mu_0 = c_0\ a_\infty/8s$

$A_1 = \mu_0/(1 + \mu_0)\ (\alpha - \alpha_0)$ (16.15)

$A_1 = C_L/\pi(AR)$ from Eqn (16.2)

$C_L/(\alpha - \alpha_0) = a =$

$\mu_0 = c_0\ a_\infty/8s = a_\infty/\pi(AR)$

$a = a_\infty/\{1 + [a_\infty/\pi(AR)]\}$ (16.16)

$a_\infty = a/\{1 - [a/\pi(AR)]\}$

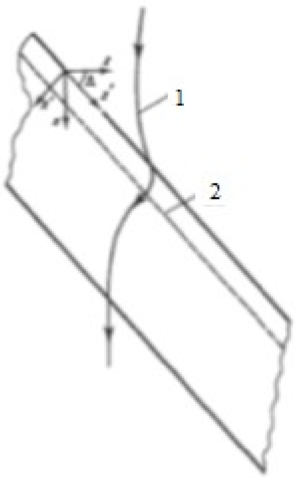

Fig. 16.6 Streamline over a sheared wing of infinite span
1 – Streamline; 2 – minimum-pressure line

$$\left(\frac{dx'}{dz'}\right)_{SL} = \frac{U_\infty \cos\Lambda + u'}{U_\infty \sin\Lambda}$$

$$L = \frac{1}{2}\rho\left(U_\infty \cos\Lambda\right)^2 S\left(\frac{dC_L}{d\alpha}\right)_{2D}\left(\alpha_n - \alpha_{0n}\right)$$

$$\alpha_n = \alpha/\cos\Lambda$$

$$\frac{dC_L}{d\alpha} = \left(\frac{dC_L}{d\alpha}\right)_{2D}\cos\Lambda \simeq 2\pi\cos\Lambda$$

$$L \propto \cos\Lambda$$

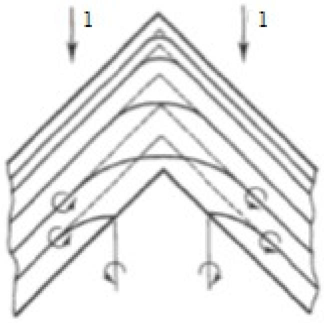

Fig. 16.7 Vortex sheet model for a swept wing 1 – flow

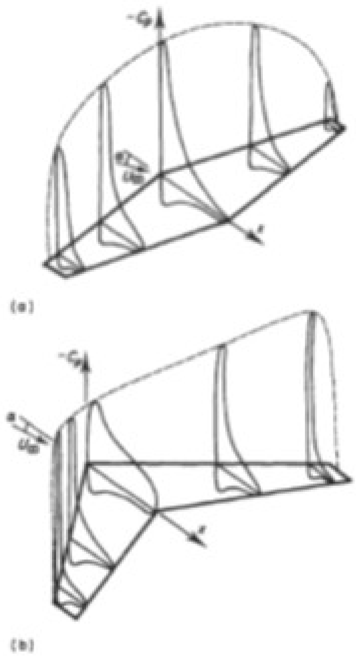

Fig. 16.8 A comparison between the pressure distributions over straight and swept-back wings

Control questions
1. Describe the typical distributions of spanwise lift.
2. Describe the load by the selected sinusoids.
3. Give the description of aerodynamic characteristics at the general symmetrical loading.
4. Describe the condition of minimum inductive drag.
5. Describe the definition of load distribution on given wing.
6. Give a description of the general theory of wings with great span.
7. Describe the load distribution for minimum drag.
8. Describe the aerodynamics of deflected wings of infinite span.n
9. Describe the aerodynamics of the swept-back wings of the final span.

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Lecture 17. Compressible flow

$p_1/\rho_1 T_1 = p_2/\rho_2 T_2$, $p_1/\rho^{\gamma}_1 = p_2/\rho^{\gamma}_2$

$T_2/T_1 = (\rho_2/\rho_1)^{\gamma-1} = (p_2/p_1)^{(\gamma-1)/\gamma}$

$d(\rho u A)/dx = 0$ (17.1)

$d(\rho u^2 A)/dx\ A\ dp/dx = 0$ (17.2)

$d(c_p T + u^2/2)/dx = 0$

$d(p/\rho T)/dx = 0$ (17.3)

$d\rho/\rho + du/u + dA/A = 0$ (17.4)

$dp/p - d\rho/\rho - dT/T = 0$ (17.5)

$\rho u A\ du + A\ dp = 0$

$dp/p = -(\gamma - 1) M^2\ du/u$

$dT/T = -(\gamma - l) M^2\ du/u$ (17.6)

$(M^2 - 1)\ du/u = dA/A$ (17.7)

$p_1/\rho_1 T_1 = p_2/\rho_2 T_2$
$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$
$\rho_1 u^2_1 A_1 - \rho_2 u^2_2 A_2 + p_1 A_1 - p_2 A_2 + \frac{1}{2}(p_1 + p_2)(A2 - A_1) = 0$
$c_P T_1 + u^2_1/2 = c_P T_2 + u^2_2/2$

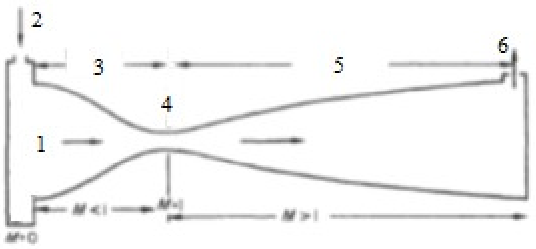

Fig. 17.1 One-dimensional isentropic expansive flow
1 – reservoir or stagnation condition ρ_0, p_0 etc; 2 – from compressor; 3 - subsonic flow; 4 – throat or sonic condition ρ^*, p^* etc; 5 - supersonic flow; 6 – to vacuum pump

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} = \sqrt{(\gamma - 1) c_P T} = u/M$$

$$\left(\begin{array}{c} \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \\ \frac{u_1^2}{2} + \frac{a_1^2}{\gamma - 1} = \frac{u_2^2}{2} + \frac{a_2^2}{\gamma - 1} \\ ¿ \\ ¿\, M_1^2 + \frac{2}{\gamma - 1} = \left(M_1^2 + \frac{2}{\gamma - 1}\right)\left(\frac{a_2}{a_1}\right)^2 \end{array} \right)$$

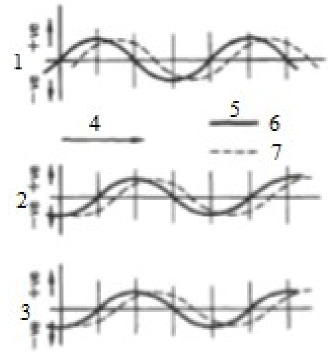

Fig. 17.2 *T*ypical set of curves for the passage of small pressure impulses
1 – displacement; 2 – velocity; 3 – pressure; 4 - Direction of wave propagation; 5 - Values for successive particles in direction of wave motion; 6 – at constant t; 7 – at constant t + δt

$u\, \partial\rho/\partial x + \rho\, \partial u/\partial x = 0$

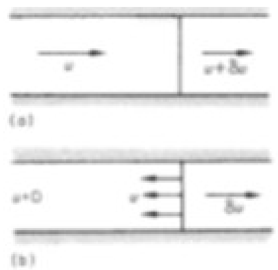

Fig. 17.3 The pressure pulse passes the stream tube
(a) Stationary wave; (b) Moving wave

$u\, \partial u/\partial x = - 1/\rho\, \partial\rho/\partial x$

$u\, \partial u/\partial x = u\, \partial u/\partial x = \partial p/\partial\rho$

$a = \sqrt{(\partial p/\partial\rho)} = \sqrt{(\gamma p/\rho)}$

$p_1/\rho_1 T_1 = p_2/\rho_2 T_2$

$m' = \rho_1 u_1 = \rho_2 u_2$

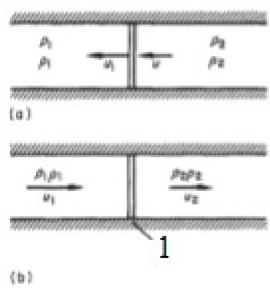

Fig. 17.4 The flow model in which a shock advances
1 - Stationary shock

$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$c_p\, T_1 + u_1^2/2 = c_p\, T_2 + u_2^2/2 = c_p\, T_0$

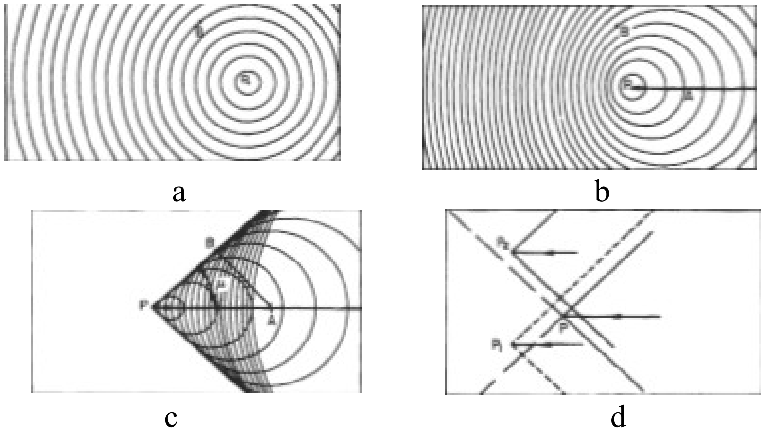

Fig. 17.5 The wave pattern associated with a point source P of weak pressure disturbances
a) Stationary source P
B represents position of wave front t sec after emission
PB = at
HB All fluid is eventually disturbed
b) Source moving at subsonic velocity $u<a$
B=position of wave front t sec after emissim from A
AB= at
PA displacement of P in t sec
PA= at
HB All fluid is eventually disturbed

c) Source moving at supersonic speed $u > a$
B=position of wave front t sec after emission from A
AB= at
PA=displacement of P in t sec
PA= ut
HB Disturbed fluid confined within Mach wedge (or cone)

d) P_1 is in the 'forward image' of the Mach wedge (or cone) of P and consequently P is within the Mach wedge of P_1 (dashed)
P_2 is outside and cannot affect P with its Mach wedge (full line)

μ = arc sin at/ut = arc sin 1/M

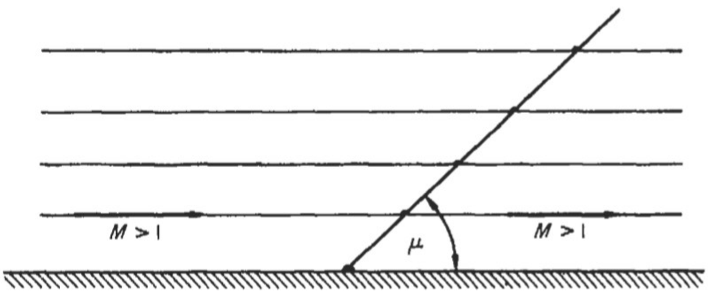

Fig. 17.6 The Mach wave emanating from a disturbance

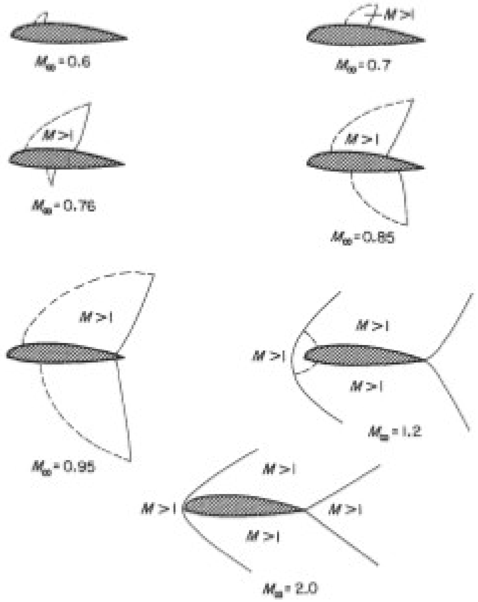

Fig. 17.7 Flow development on two-dimensional aerofoil as M_∞ increases beyond M_{crit}; $M_{crit} = 0.58$

Рис. 17.7 Розвиток течії на двомірному профілі при збільшенні M_∞ за межі M_{crit}; $M_{crit} = 0,58$

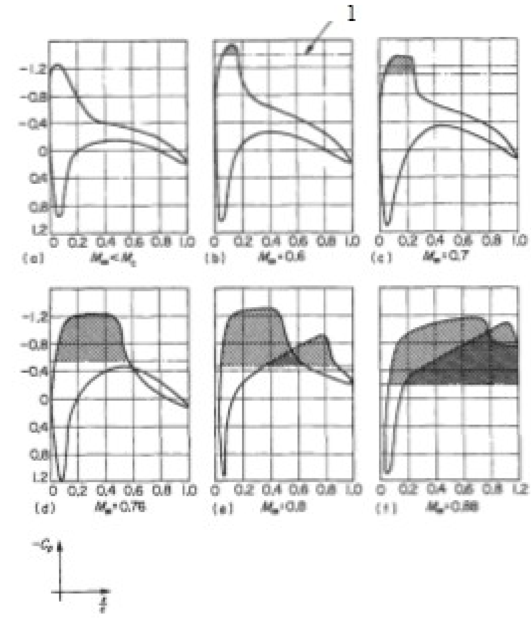

Fig.17.8 Pressure distribution on two-dimensional aerofoil ($M_{crit} = 0.57$) as M_∞ increases through M_{crit}.
1 — Local Mach number = 1

Рис.17.8 Розподіл тиску на двомірному профілі ($M_{crit} = 0,57$) при збільшенні M_∞ через M_{crit}.
1 — місцеве число Маха = 1

$C_{pmin} = (p_{min} - p_\infty) / (\frac{1}{2}\rho_\infty V_\infty^2)$ (17.7)

$C_{pmin} = [p_{min} / p_\infty - 1]\, 2/ (\gamma M_\infty^2)$

$C_{pcrit} = [p^* / p_\infty - 1]\, 2/ (\gamma M_{crit}^2)$ (17.8)

$$\frac{M_{crit}^2}{2}+\frac{1}{\gamma-1}=\left(\frac{a^*}{a_\infty}\right)\left(\frac{1}{2}+\frac{1}{\gamma-1}\right)$$

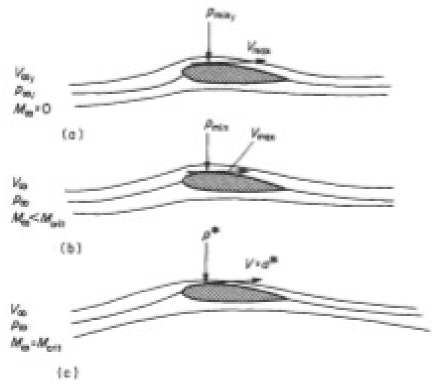

Fig. 17.9 (a) 'Incompressible' flow. (b) Compressible subscritical flow. (c) Critical flow

Рис. 17.9 Течія (a) "Нестислива", (b) Стислива докритична, (c) Критична

$$\left(\frac{a^*}{a_\infty}\right)^2=M_{crit}^2\frac{\gamma-1}{\gamma+1}+\frac{2}{\gamma+1}$$

$$\frac{p^*}{p_\infty}=\left(\frac{a^*}{a_\infty}\right)^{2\gamma/(\gamma-1)}$$

$$C_{p_{crit}}=\left[\left(\frac{\gamma-1}{\gamma+1}M_{crit}^2+\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}-1\right]\frac{2}{\gamma M_{crit}^2}$$

(17.9)

$$C_{p_{crit}}=\frac{C_{p_0}}{\sqrt{1-M_{crit}^2}}$$

(17.10)

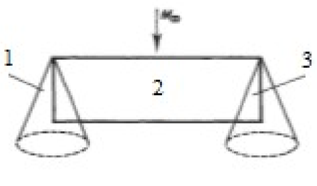

Fig. 17.10 A typical wing with a supersonic leading edge

1 - Mach cone; 2 – Two-dimensional flow; 3 - Tip effects

Рис. 17.10 Типове крило із надзвуковою передньою крайкою

1 — конус Маха; 2 — двомірна течія; 3 - Ефекти закінцівок

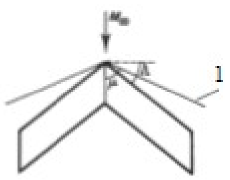

Fig. 17.11 A wing with a subsonic leading edge

Рис. 17.11 Крило з дозвуковою передньою крайкою

1 - Mach cone

1 — конус Маха

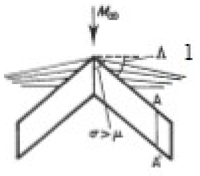

Fig. 17.12 A shock cone generated with wings having finite thickness and incidence
1 – successive chock waves

Рис. 17.12 Ударний конус генерований крилами з кінцевою товщиною і кутом атаки
1 – послідовні хвилі ущільнення

Контрольні питання

1. Дайте опис адіабатичної одномірної течії і наведіть рівняння збереження та стану для квазіодномірної, адіабатичної течії в диференційній формі (зокрема, розширюваної течії).

2. Враховуючи різні способи, наведіть рівняння, в яких швидкість звуку може бути виражена в адіабатичній течії.

3. Дайте опис слабких стрибків ущільнення в одномірній течії.

4. Дайте опис швидкості течії в трубці течії.

5. Дайте опис плоских прямих стрибків ущільнення в одномірній течії.

6. Дайте опис стрибків Маха і стрибків ущільнення в двомірній течії.

7. Дайте опис крил у стисливій течії.

8. Дайте опис критичного коефіцієнту тиску у стисливій течії.

9. Дайте опис крил кінцевого розмаху у стисливій течії.

Recommended literature

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